UNIT – IV
FREQUENCY RESPONSE ANALYSIS

Topics: Introduction to frequency domain specifications – Bode diagrams – transfer function from the Bode diagram –Polar plots, Nyquist stability criterion- stability analysis using Bode plots (phase margin and gain margin).
Classical Control Design Techniques: Lag, lead, lag-lead compensators - physical realization - design of compensators using Bode plots.

INTRODUCTION:

The frequency response is the steady state response (output) of a system when the input to the system is a sinusoidal signal. i.e. It is the magnitude and phase relationship between sinusoidal input and steady state output of a system.

In the system transfer function T(s), if the **'s'** is replaced by **'jω'** then the resulting transfer function T(jω) is called sinusoidal transfer function. The frequency response of the system is directly obtained from the sinusoidal transfer function T(jω) of the system. The transfer function T(jω) is a complex function of frequency.

The magnitude and phase of T(jω) are the functions of frequency and can be evaluated for various values of frequency.

Let

$$T(s) = \frac{C(s)}{R(s)}$$

Put s = jω

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = M\angle\emptyset$$

Where M = Magnitude of |T(jω)|

Φ = Phase of ∠T(jω)

ADVANTAGES OF FREQUENCY RESPONSE ANALYSIS

(i) Analytically, it is more difficult to determine the time response of the system for higher order systems.

(ii) As there exists numerous ways of designing a control system to meet the time domain performance specifications, it becomes difficult for the designer to choose a suitable design for a particular system.

(iii) The transfer function of a higher order system can be identified by computing the frequency response of the system over a wide range of frequencies ω.

(iv) The time-domain specifications of a system can be met by using the frequency domain specifications as a correlation exists between the frequency response and time response of a system.

(v) The stability of a non-linear system can be analysed by the frequency response analysis.

(vi) The transfer function of a higher order system can be obtained using frequency response analysis which makes use of physical data when it is difficult to obtain using differential equations.

(vii) The frequency response analysis can be applied to the system that has no rational transfer function (i.e., a system with transportation lag).
(viii) The frequency response analysis can be applied to the system even when the input is not deterministic.
(ix) The frequency response analysis is very convenient in measuring the system sensitivity to noise and parameter variations.
(x) In frequency response analysis, stability and relative stability of a system can be analysed without evaluating the roots of the characteristic equation of the system.
(xi) The frequency response analysis is simple and accurate.

DISADVANTAGES OF FREQUENCY RESPONSE ANALYSIS

(i) Frequency response analysis is not recommended for the system with very large time constants.
(ii) It is not useful for non-interruptible systems.
(iii) It can generally be applied only to linear systems. When this approach is applied to a non-linear system, the result obtained is not exact.
(iv) It is considered as outdated when compared with the methods developed for digital computer and modelling.

PLOTTING OF FREQUENCY RESPONSE

Frequency Response = Magnitude Response + Phase Response
= $|G(j\omega)| + \angle G(j\omega)$ [ω is from 0 to ∞]

EXAMPLE: Draw the frequency response for $G(s) = \frac{1}{1+2s}$
SOL:

The sinusoidal transfer function, $G(j\omega) = \frac{1}{1+j2\omega}$

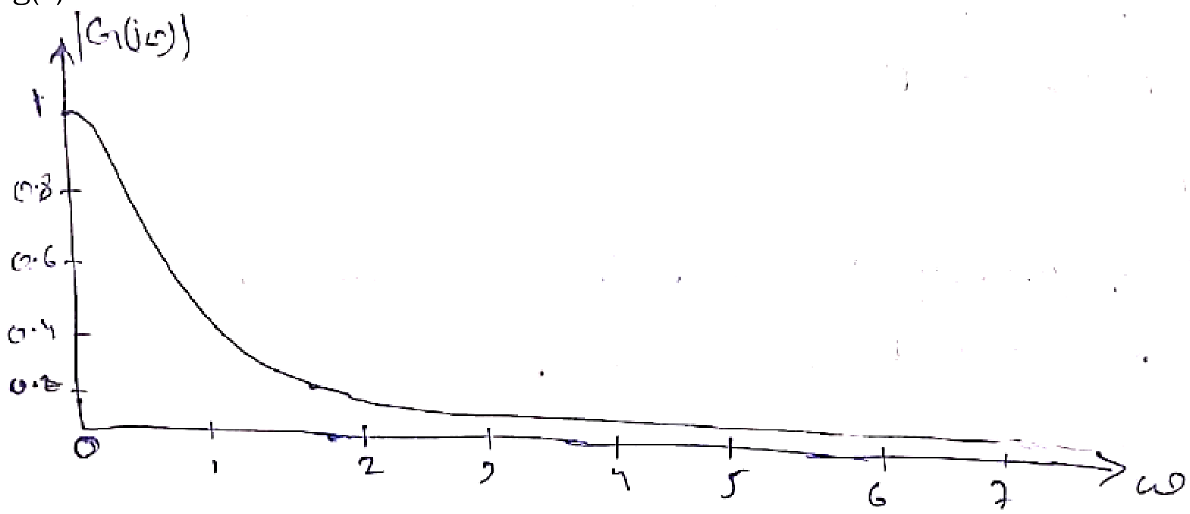
The magnitude response is $|G(j\omega)| = \frac{1}{\sqrt{1+4\omega^2}}$

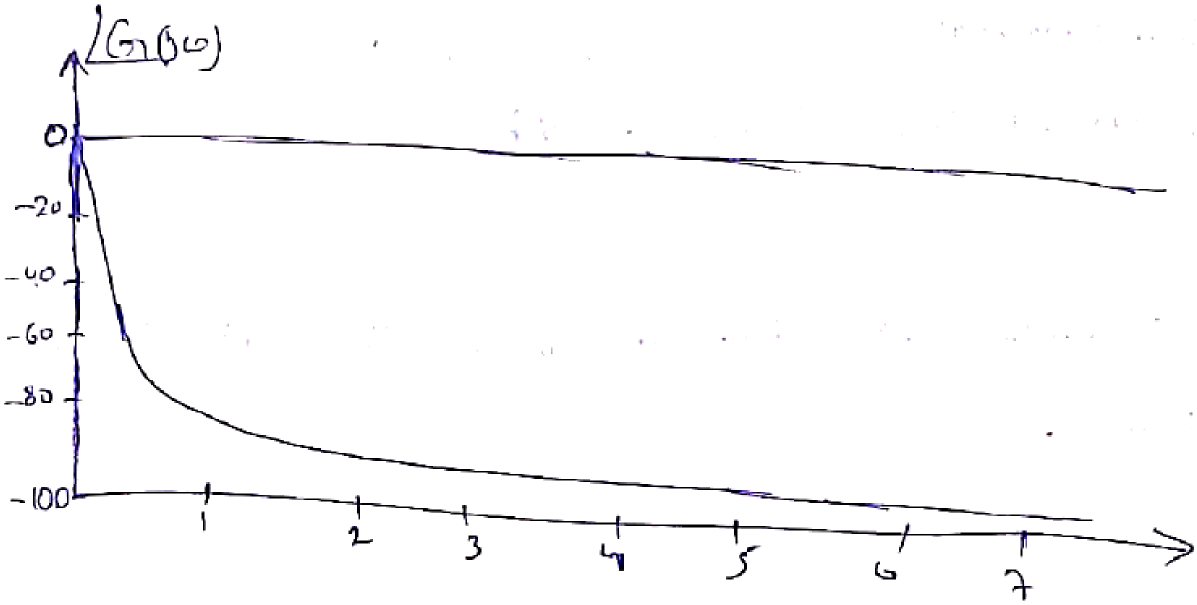
The Phase response is $\angle G(j\omega) = -\tan^{-1}(2\omega)$

Now vary the frequency ω from 0 to ∞ and tabulate the values

ω	0	1	2	5	10	20	50	100	∞
$\lvert G(j\omega)\rvert$	1	0.4472	0.2423	0.099	0.05	0.025	0.01	0.005	0
$\angle G(j\omega)$	0^0	-63.43^0	-75.96^0	-84.29^0	-87.13^0	-88.56^0	-89.42^0	-89.7^0	-90^0

The magnitude response and phase response are shown in the following fig(s).





FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications.

The frequency domain specifications are

1) Resonant Peak (M_r)
2) Resonant Frequency (ω_r).
3) Bandwidth
4) cut-off Frequency
5) Cut-off Rate
6) Gain-Margin
7) Phase-Margin.

Resonant Peak (M_r)

The max. value of the magnitude of closed loop T.F is called resonant Peak (M_r).

$$\text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Resonant Frequency (ω_r)

The frequency at which the resonant peak occurs is called Resonant Frequency (ω_r).

$$\text{The resonant frequency, } \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Band width (ω_b)

The Bandwidth is the range of frequencies for which the system gain is more than $-3db$.

$$\therefore \boxed{\text{Bandwidth, } \omega_b = \omega_n\, u_b = \omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{\frac{1}{2}}}$$

Cut-off Frequency

The frequency at which the gain is $-3db$ is called cut-off frequency.

Cut-off Rate

The slope of log-magnitude curve near the cut-off frequency is called cut-off rate.

Gain Margin (Kg)

The gain Margin (kg) is defined as the reciprocal of the magnitude of open loop T.F. at phase cross over frequency.

$$\text{Gain Margin, } Kg = \frac{1}{|G(j\omega_{pc})|} \Rightarrow Kg \overset{\text{in db}}{=} 20\log kg = 20\log\frac{1}{|G(j\omega_{pc})|}$$

$$kg = -20\log|G(j\omega_{pc})|\, db$$

Phase cross over Frequency (ω_{pc})

The Frequency at which the phase of open loop T.F. is 180° is called phase crossover frequency (ω_{pc}).

Phase Margin (γ)

The phase margin is obtained by adding 180° to the phase angle ϕ of the open loop T.F. at gain cross over freq. (ω_{gc})

phase Margin $\gamma = 180 + \phi_{gc}$

$\Rightarrow \phi_{gc} = \angle G(j\omega_{gc})$

Gain cross over Frequency (ω_{gc})

The Frequency at which the magnitude of the open loop T.F is unity (db = 0) is called Gain cross over Frequency (ω_{gc}).

PROBLEMS

1) The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{81}{s(s+8)}$$

Determine the resonant frequency and resonant peak for the system.

SOL:

Given $G(s) = \frac{81}{s(s+8)}$ and $H(s) = 1$.

Hence, the closed-loop transfer function of the system, $\frac{C(s)}{R(s)} = \frac{81}{s^2+8s+81}$

The characteristic equation of the given system is $s^2+8s+81=0$

Comparing the above equation with the standard second-order characteristic $s^2+2\xi\omega_n s+\omega_n^2=0$, we obtain

$$\omega_n^2 = 81, \text{ i.e., } \omega_n = 9 \text{ rad/sec and } 2\xi\omega_n = 8, \text{ i.e., } 2\xi \times 9 = 8$$

Hence, $\xi = 0.44$

$$\text{Resonant peak, } M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.44\sqrt{1-(0.44)^2}} = 1.265$$

$$\text{Resonant frequency, } \omega_r = \omega_n\sqrt{1-2\xi^2} = 9\sqrt{1-2\times(0.44)^2} = 7.045 \text{ rad/sec}$$

2) The closed loop poles of a system are at s = –2 ± j3.
Determine i) Bandwidth ii) Normalized peak driving signal frequency
iii) Resonant peak for the system.

SOL:

The closed-loop transfer function of any system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

and the characteristic equation of the system is given by

$$1+G(s)H(s)=0$$

The closed-loop poles of a system are obtained by equating the characteristic equation to zero. Hence, the characteristic equation of the given system using the given closed-loop poles is obtained as $(s+2-j3)(s+2+j3)=0$

i.e., $s^2+4s+13=0$

Comparing the above equation with the standard second-order characteristic equation, we obtain $\omega_n^2 = 13$. Therefore, $\omega_n = 3.605$ rad/sec

$$2\xi\omega_n = 4$$

Therefore, $$\xi = \frac{2}{\omega_n} = 0.5547$$

(i) Bandwidth (BW) = $\omega_n\sqrt{1-2\xi^2 \pm \sqrt{(1-2\xi^2)^2+1}}$

$$= 3.605 \times \sqrt{1-\left(2\times 0.5547^2\right) \pm \sqrt{\left(1-\left(2\times 0.5547^2\right)\right)^2+1}} = 4.350 \text{ rad/sec}$$

(ii) Normalized peak driving signal frequency (u_p) is

$$u_p = \sqrt{1-2\xi^2} = \sqrt{1-2(0.5547)^2} = 0.3846$$

(iii) Resonant peak, $M_p = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

Therefore, $$M_p = \frac{1}{2\times 0.5547\times\sqrt{1-(0.5547)^2}} = 1.083$$

3) Determine the frequency specification of a second order system whose closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{64}{s^2+10s+64)}$$

SOL:

Comparing denominator of the transfer function with $s^2+2\xi\omega_n s+\omega_n^2$, we obtain

$$\omega_n^2 = 64 \text{ i.e., } \omega_n = 8 \text{ and } 2\xi\omega_n = 10 \text{ i.e., } \xi = 0.625$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.625\times\sqrt{1-0.625^2}} = 1.0248$$

and $$\omega_r = \omega_n\sqrt{1-2\xi^2} = 8\times\sqrt{1-2(0.625)^2} = 3.741 \text{ rad/sec}$$

$$BW = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = 8\sqrt{1-2(0.625)^2+\sqrt{2-4(0.625)^2+4(0.625)^4}} = 8.917 \text{ rad/sec}$$

GRAPHICAL REPRESENTATION OF FREQUENCY RESPONSE

Determining the frequency response of a system, i.e., the magnitude and phase angle of a system for different frequencies from 0 to ∞ by using tabulation method becomes more complicated when more number of poles and zeros exist in the system. An alternative method that eliminates the difficulty of the tabulation method is the graphical representation of frequency response.

There are different graphical methods by which the frequency response can be represented. They are

(i) Bode plot (asymptotic plots)
(ii) Polar plot
(iii) Nyquist plot
(iv) Constant *M* and *N* circles
(v) Nichols chart

BODE PLOT

The Bode Plot is a graphical representation of the T.F for determining the stability of the system. The Bode Plot can be drawn for both open loop & closed loop TF. Usually the Bode Plot is drawn for open loop system.

The Bode plot consists of two seperate plots.

1) The Plot of the magnitude of sinusoidal T.F Vs $\log\omega$
2) The plot of the phase angle of sinusoidal T.F Vs $\log\omega$

The curves are drawn on semi log graph paper.

i.e. The two plots are 1) $20\log_{10}|G(j\omega)|$ Vs $\log\omega$
2) Phase angle Vs $\log\omega$

The main Advantage of using Bode Plot is that multiplication of magnitudes can be converted into Addition.

Consider the open loop transfer function, $G(s)=\dfrac{K\,(1+sT_1)}{s\,(1+sT_2)\,(1+sT}$

$$G(j\omega)=\frac{K\,(1+j\omega T_1)}{j\omega\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$=\frac{K\angle 0^\circ\ \sqrt{1+\omega^2T_1^2}\ \angle\tan^{-1}\omega T_1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\ \angle\tan^{-1}\omega T_3}$$

The magnitude of $G(j\omega)=\ |G(j\omega)|=\dfrac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\ \sqrt{1+\omega^2T_3^2}}$

The phase angle of the $G(j\omega)=\angle G(j\omega)=\tan^{-1}\omega T_1-90^\circ-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3$

The magnitude of G(jω) can be expressed in decibels as shown below.

$$|G(j\omega)| \text{ in db} = 20 \log |G(j\omega)|$$

$$= 20 \log \left[\frac{K \sqrt{1+\omega^2 T_1^2}}{\omega \sqrt{1+\omega^2 T_2^2} \sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \left[\frac{K}{\omega} \times \sqrt{1+\omega^2 T_1^2} \times \frac{1}{\sqrt{1+\omega^2 T_2^2}} \times \frac{1}{\sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_2^2}} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_3^2}}$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} - 20 \log \sqrt{1+\omega^2 T_2^2} - 20 \log \sqrt{1+\omega^2 T_3^2}$$

From this eq., when the magnitude is expressed in db, the multiplication is converted into addition.

Hence the magnitude plot, the db magnitudes of individuals factors G(jω) H(jω) can be added.

The magnitude plot and phase plot of various factors of G(jω) H(jω) are explained below. that are frequently occuring

BASIC FACTORS OF G(jω)

The basic factors that very frequently occur in a typical transfer function G(jω) are,

1. Constant gain, K
2. Integral factor, $\frac{K}{j\omega}$ or $\frac{K}{(j\omega)^n}$
3. Derivative factor, $K \times j\omega$ or $K \times (j\omega)^n$
4. First order factor in denominator, $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. First order factor in numerator, $(1 + j\omega T)$ or $(1 + j\omega T)^m$
6. Quadratic factor in denominator, $\left[\frac{1}{1+2\zeta (j\omega / \omega_n) + (j\omega / \omega_n)^2}\right]$
7. Quadratic factor in numerator, $\left[1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$

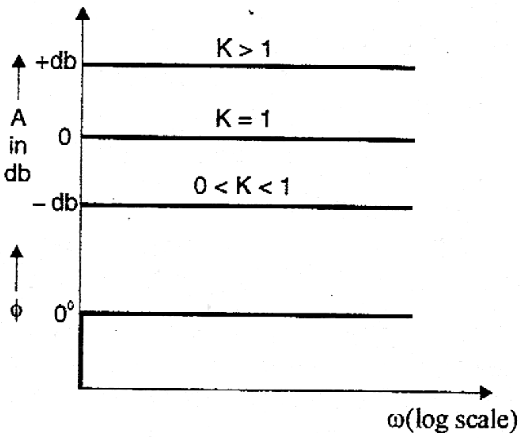
Constant Gain, K

Let, $G(s) = K$

$$\therefore G(j\omega) = K = K\angle 0°$$

$$A = |G(j\omega)| \text{ in db} = 20 \log K$$

$$\phi = \angle G(j\omega) = 0°$$



The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20 log K db. The phase plot is straight line at 0°.

When $K > 1$, 20 log K is positive.

When $0 < K < 1$, 20 log K is negative.

When $K = 1$, 20 log K is zero.

Integral Factor

Let, $G(s) = \dfrac{K}{s}$

$$\therefore G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega}\angle -90°$$

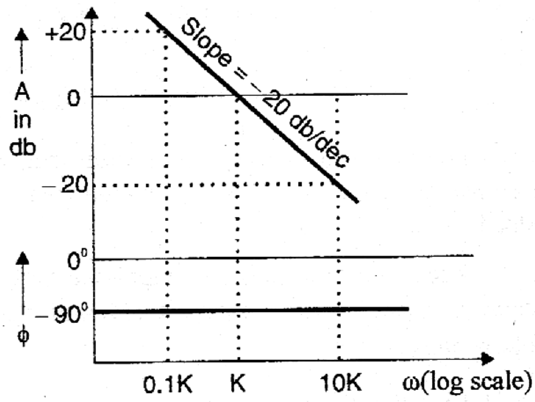
$$A = |G(j\omega)| \text{ in db} = 20 \log (K/\omega)$$

$$\phi = \angle G(j\omega) = -90°$$

When $\omega = 0.1\,K$, $A = 20 \log (1/0.1) = 20$ db

When $\omega = K$, $A = 20 \log 1 = 0$ db

When $\omega = 10\,K$, $A = 20 \log (1/10) = -20$ db



From the above analysis it is evident that the magnitude plot of the integral factor is a straight line with a slope of –20 db/dec and passing through zero db, when $\omega = K$. Since the $\angle G(j\omega)$ is a constant and independent of ω the phase plot is a straight line at –90°.

When an integral factor has multiplicity of n, then,

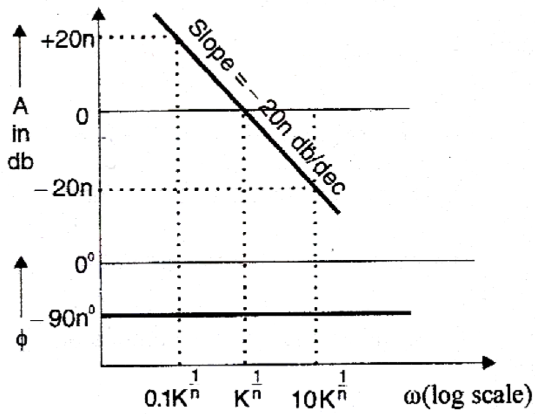
$$G(s) = K/s^n$$

$$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n°$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{K}{\omega^n}$$

$$= 20\log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)^n = 20\, n \log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90\, n°$$



Now the magnitude plot of the integral factor is a straight line with a slope of –20n db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at –90n°.

Derivative Factor

Let, $G(s) = Ks$

$$\therefore G(j\omega) = K\,j\omega = K\omega\ \angle 90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20\log(K\omega)$$

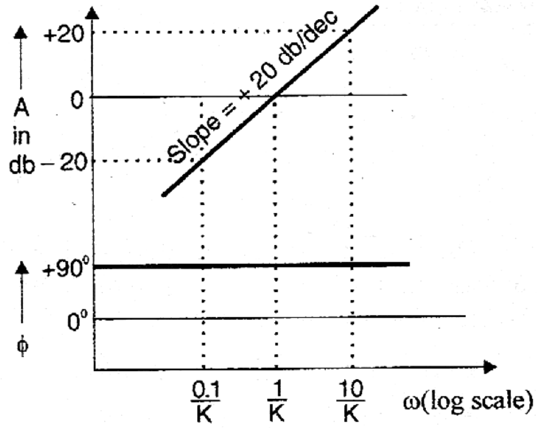
$$\phi = \angle G(j\omega) = +90^\circ$$

When $\omega = 0.1/K$, $A = 20\log(0.1) = -20$ db

When $\omega = 1/K$, $A = 20\log 1 = 0$ db

When $\omega = 10/K$, $A = 20\log 10 = +20$ db

From the above analysis it is evident that the magnitude plot of the derivative factor is a straight line with a slope of +20 db/dec and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at $+90^\circ$.



When derivative factor has multiplicity of n then,

$$G(s) = K\,s^n$$

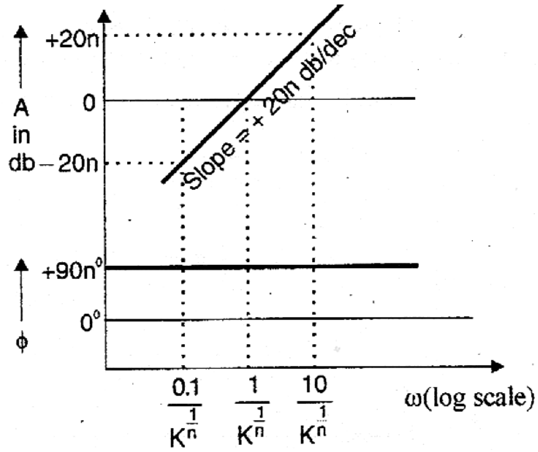
$$\therefore G(j\omega) = K(j\omega)^n = K\omega^n\ \angle 90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20\log(K\omega^n)$$

$$= 20\log(K^{1/n}\omega)^n = 20\,n\log(K^{1/n}\omega)$$

$$\phi = \angle G(j\omega) = 90n^\circ$$

Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at $+90n^\circ$.



First order factor in denominator

$$G(s) = \frac{1}{1+sT}$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}}\ \angle -\tan^{-1}\omega T$$

Let, $A = |G(j\omega)|$ in db.

$$\therefore\ A = |G(j\omega)|_{\text{in db}} = 20\log\frac{1}{\sqrt{1+\omega^2T^2}} = -20\log\sqrt{1+\omega^2T^2}$$

At very low frequencies, $\omega T \ll 1$; $\quad \therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log 1 = 0$

At very high frequencies, $\omega T \gg 1$; $\quad \therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log\sqrt{\omega^2T^2} = -20\log\omega T$

At $\omega = \frac{1}{T}$, $\quad A = -20\log 1 = 0$

At $\omega = \frac{10}{T}$, $\quad A = -20\log 10 = -20$ db i.e. The magnitude plot consists of two straight lines i.e. at 0db & the other is with slope of -20 db/dec. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called corner frequency (ω_c).

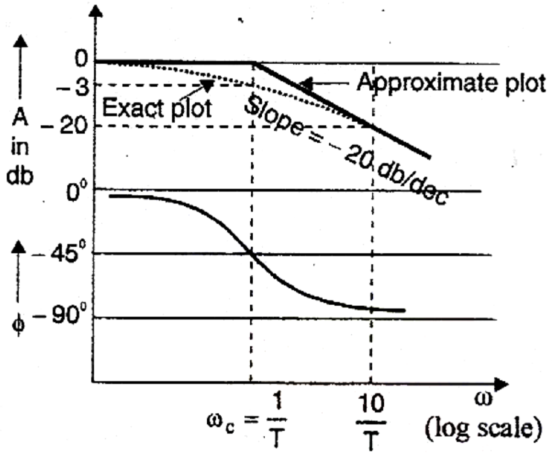
phase angle $\phi = -\tan^{-1}\omega T$

at $\omega = \frac{1}{T} = \omega_c$, $\phi = -\tan^{-1} 1 = -45°$

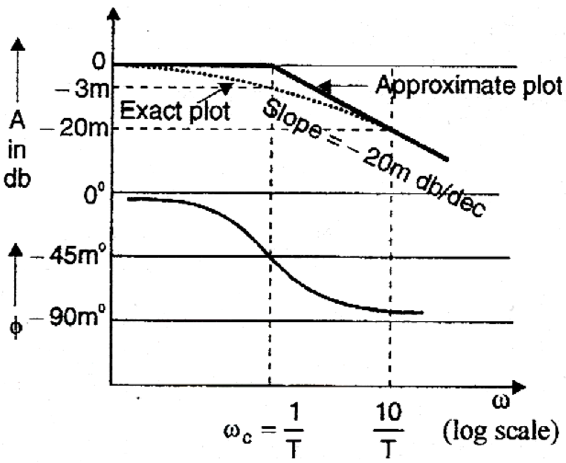
$\omega = 0$, $\phi = 0$

$\omega = \infty$, $\phi = -90°$

The Bode plot for $G(S) = \frac{1}{1+ST}$ can be shown below



Bode plot of the factor $\frac{1}{1+j\omega T}$.



Bode plot of the factor $1/(1+j\omega T)^m$.

FIRST ORDER FACTOR IN THE NUMERATOR

Let $G(S) = 1+ST$

$G(j\omega) = 1+j\omega T = \sqrt{1+\omega^2T^2} \angle \tan^{-1}\omega T$

$M = |G(j\omega)|$ in db $= 20 \log\sqrt{1+\omega^2T^2}$

$\omega T \ll 1$, $M = 20 \log 1 = 0$ db

$\omega T \gg 1$, $M = 20 \log \omega T$

at $\omega = \frac{1}{T} = \omega_c$, $M = 0$ db

$\omega = \frac{10}{T}$, $M = 20$ db

i.e. The magnitude plot consist of two straight lines (asymtotes). One is at 0db and other is with a slope of +20db/dec.

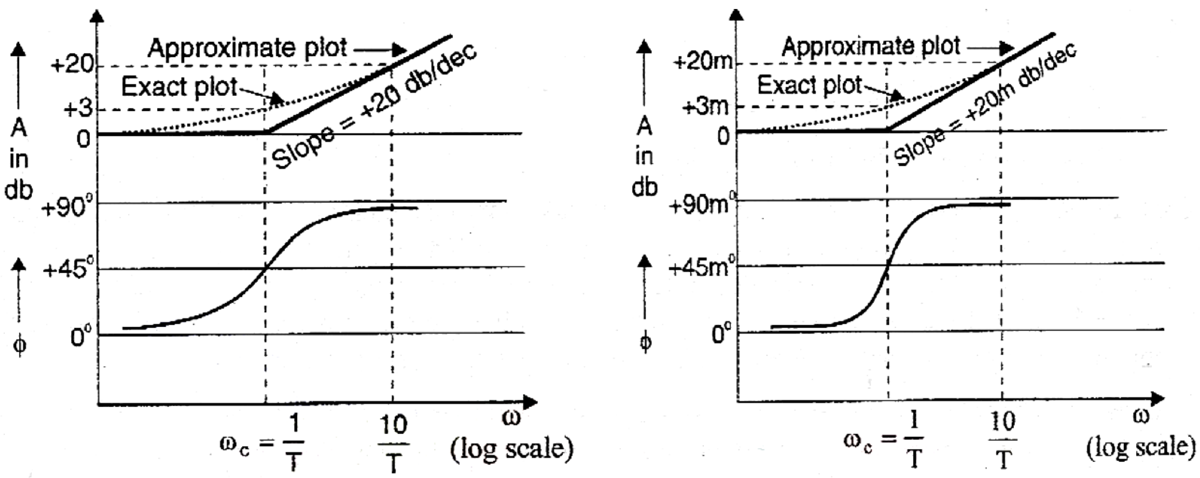
The phase angle $\phi = \tan^{-1}\omega T$

$\omega = 0$, $\phi = 0$

$\omega = \frac{1}{T}$ $\phi = 45°$

$\omega = \infty$ $\phi = 90°$

The Bode plot of $G(s) = 1+sT$ can be shown below.



Bode plot of the factor (1+jωT). *Bode plot of the factor $(1+j\omega T)^m$.*

QUADRATIC FACTOR IN THE DENOMINATOR

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{1 + 2\zeta\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2}$$

$$\therefore G(j\omega) = \frac{1}{1 + j\frac{2\zeta\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} \angle - \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Let, $A = |G(j\omega)|$ in db.

$$A = 20\log \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$= -20\log\sqrt{1+\frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 4\zeta^2\frac{\omega^2}{\omega_n^2}} = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}$$

At very low frequencies when $\omega << \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}} \approx -20\log 1 = 0$$

At very high frequencies when $\omega >> \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}} \approx -20\log\sqrt{\frac{\omega^4}{\omega_n^4}} = -20\log\frac{\omega^2}{\omega_n^2} = -20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$\therefore A = -40\log\frac{\omega}{\omega_n}$$

At $\omega = \omega_n$, $A = -40\log 1 = 0$ db

At $\omega = 10\omega_n$, $A = -40\log 10 = -40$ db

The magnitude plot consists of two straight lines one is at 0db and other is with a slope of -40 db. For quadratic factor, the frequency ω_n is the corner frequency.

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

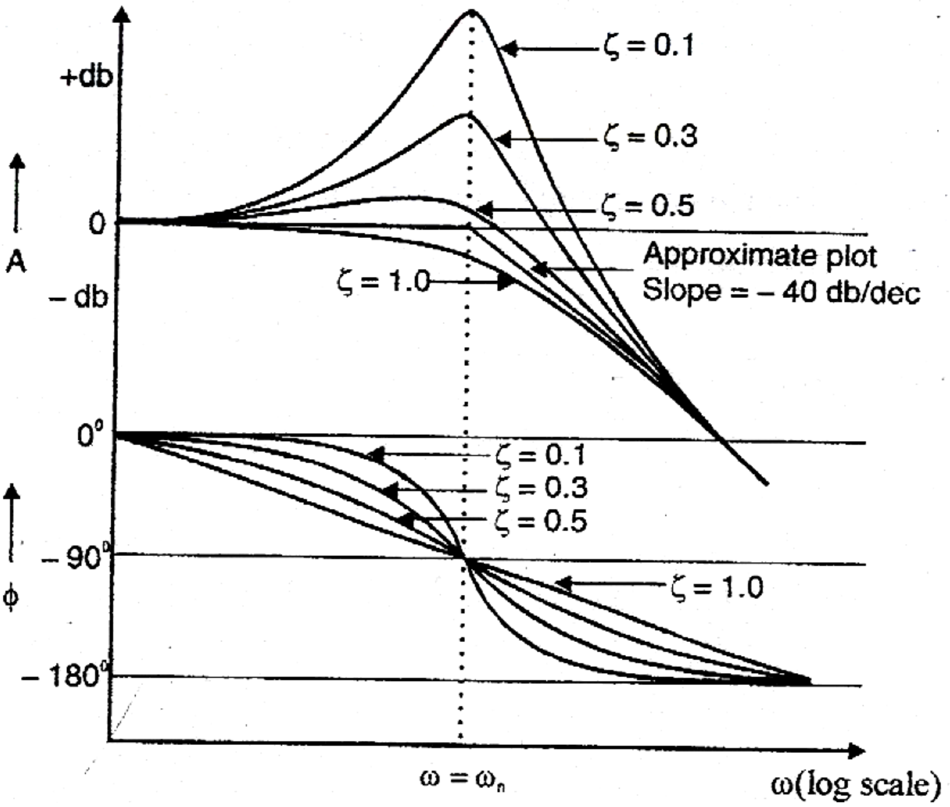
$$\text{As } \omega = \omega_n, \quad \phi = -\tan^{-1}\frac{2\zeta}{0} = -\tan^{-1}\infty = -90^\circ$$

$$\text{As } \omega \to 0, \quad \phi \to 0$$

$$\text{As } \omega \to \infty, \quad \phi \to -180^\circ$$

The phase plot is a curve passing through -90° at ω_c/ω_n.

The Bode plot can be shown below.



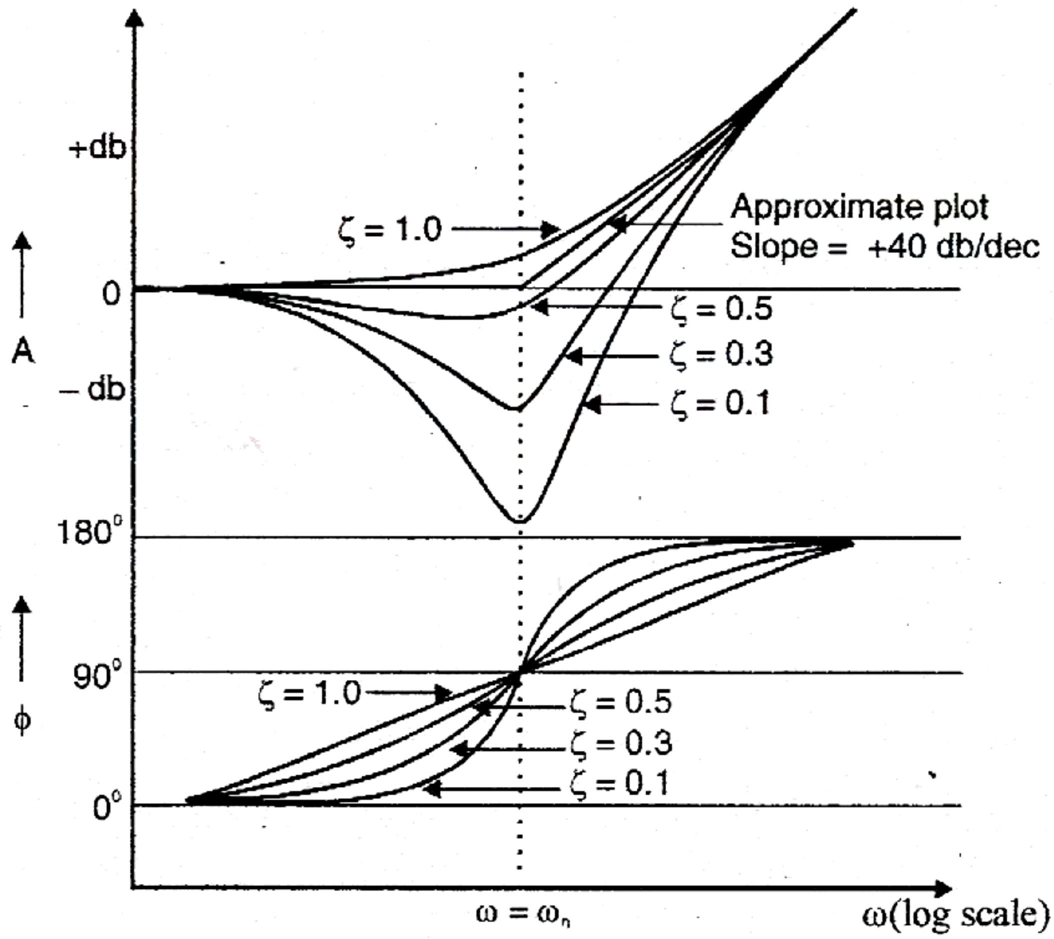
QUADRATIC FACTOR IN THE NUMERATOR

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = 1 + 2\zeta\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$G(j\omega) = 1 + j2\zeta\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}} \angle \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Based on an analysis similar to that of denominator quadratic factor, the magnitude plot of the quadratic factor in the numerator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope +40 db/dec for the frequency range $\omega_n < \omega < \infty$. The corner frequency is ω_n. Due to this approximation the error at the corner frequency depends on ζ.

The phase angle varies from 0 to +180°, as ω is varied from 0 to ∞. At the corner frequency the phase angle is +90° and independent of ζ, but at all other frequency it depends on ζ.



DETERMINATION OF GAIN MARGIN, PHASE MARGIN AND STABITY FROM BODE PLOT

Gain Margin :- Gain Margin is defined as the margin in gain allowable by which gain can be increased till system reaches on the verge of instability.

GM is defined as reciprocal of the magnitude of $G(j\omega)$ at phase crossover frequency (ω_{pc}).

$$\therefore \quad G.M. = \frac{1}{|G(j\omega_{pc})|}$$

$$G.M. \text{ in db} = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

$$= -20 \log |G(j\omega_{pc})|$$

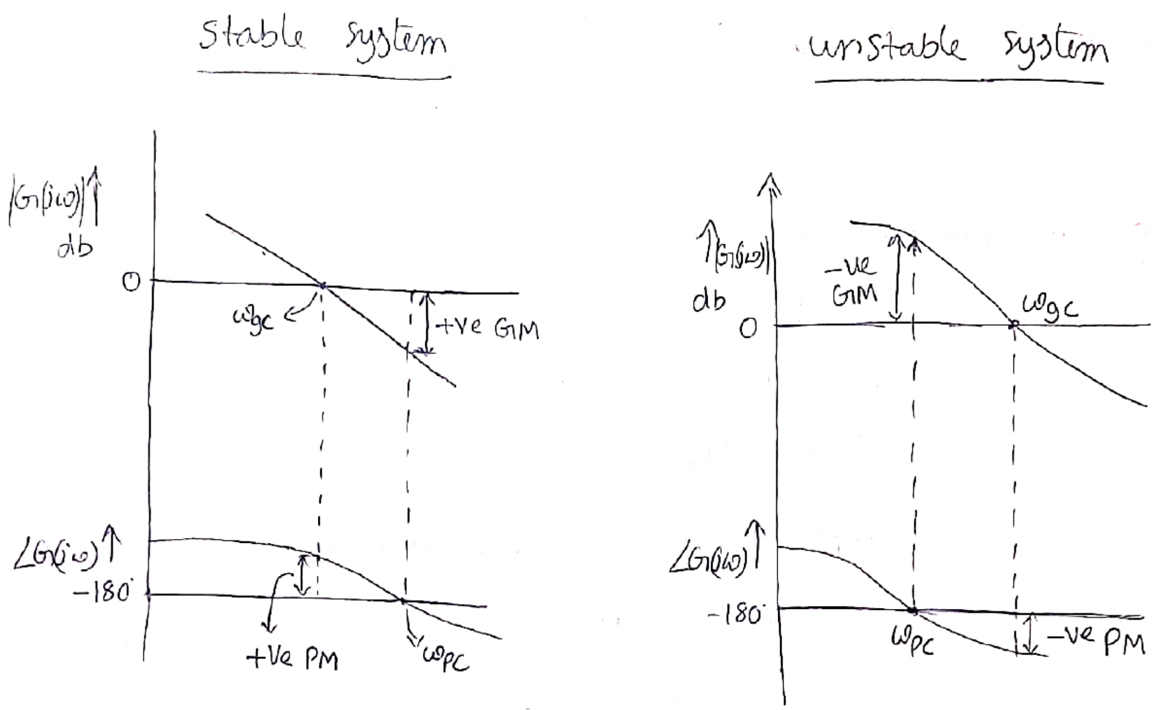
Phase Margin :- P.M. can be defined as

$$P.M = 180 + \angle G(j\omega_{gc})$$

Stability :- If the gain margin is +ve, then the system is stable.

If the GM is -ve, then the system is unstable.

The determination of GM & PM from Bode plot can be shown in the following fig.



Stability of the system based on crossover frequencies

S. No	Relation between ω_{gc} and ω_{pc}	Stability of the system
1.	$\omega_{gc} < \omega_{pc}$	Stable system
2.	$\omega_{gc} > \omega_{pc}$	Unstable system
3.	$\omega_{gc} = \omega_{pc}$	Marginally stable system

Stability of the system based on g_m and p_m

S. No.	Gain margin g_m	Phase margin p_m	Relation between ω_{gc} and ω_{pc}	Stability of the system
1	Positive	Positive	$\omega_{gc} < \omega_{pc}$	Stable system
2	Positive	Negative	$\omega_{gc} < \omega_{pc}$	Unstable system
3	Negative	Positive	$\omega_{gc} > \omega_{pc}$	Unstable system
4	0 dB	0°	$\omega_{gc} = \omega_{pc}$	Marginally stable system

PROCEDURE TO DRAW BODE PLOT

Magnitude PLOT :-

Step : 1 Convert the given Transfer function into time constant Form and then find the Sinusoidal T.F. by replacing s by $j\omega$.

Step : 2 Find the corner frequencies of each Factor in the T.F. and list them in increasing order of corner frequency. If the sinusoidal T.F. has a term like K, $\frac{K}{(j\omega)^n}$ or $K(j\omega)^n$ then enter that factor as first term in the table. Find the slope contributed by each Factor and net slope from the corner frequency. Prepare a table as shown below.

Factor	corner Frequency rad/sec	slope db/dec	change in slope (net slope) db/dec

[Note : The magnitude plot can be started with K or $\frac{K}{(j\omega)^n}$ or $K(j\omega)^n$ term and then the db magnitude of every ~~term~~ first and higher order terms are added one by one in the increasing order of corner frequency.]

step : 3 Choose an arbitrary frequency ω_l which is lesser than the lowest corner frequency. calculate the db magnitude

step : 4 Then calculate the gain (db) at every corner frequency one by one using the formula.

Gain at ω_y = change in gain from ω_x to ω_y + Gain at ω_x

$$= \left[\text{slope from } \omega_x \text{ to } \omega_y * \log \frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$

step : 5 Choose an arbitrary frequency ω_h which is greater than the highest corner frequency. calculate the gain at ω_h by using the above formula.

step : 6 In a semilog graph sheet mark the required range of frequency on x-axis and the range of db magnitude on y-axis after choosing proper scale.

step : 7 Mark all points obtained in steps 3,4 & 5 on the graph and Join the points by straight lines. Mark the slope at every part of the graph.

Phase PLOT

step : 8 To draw Phase plot, find $\angle G(j\omega)$ from sinusoidal T.F. Vary ω for entire range of frequency scale and find $\angle G(j\omega)$ and tabulate.

ω	
$\angle G(j\omega)$	

Mark all these points on the semilog graph and draw a smooth curve to get Phase plot.

[Note :- The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog sheet on a common frequency scale.]

DETERMINATION OF TRANSFER FUCNTION FROM BODE PLOT

Transfer function of a system can be obtained from its experimental data if one can plot the Bode diagram from the experimental data.

The simple rules to get the different factors of the transfer function from experimental Bode plot are as follows.

1) The system type can be determined from the slope of Bode plot at low frequencies (left most part)

Low Frequency slope	Type
0 db/dec	Type 0
−20 db/dec	Type−1
−40 db/dec	Type−2

If the low frequency asymptote is a horizontal line through A db, then the transfer function represents a type-0 system with a system gain k given by

$$A = 20\log k$$

If the low frequency asymptote has a slope of $-20db/dec$ then the Transfer function has a factor of the form $\frac{K}{S}$.

If the low frequency asymptote has a slope of $-40db/dec$ then the Transfer function has a factor of the form $\frac{K}{S^2}$.

2) A change in slope at a frequency indicates the presence of another factor. If the change in slope at $\omega = \omega_{c_1}$ is $-20\, db/dec$, it indicates the presence of a first order factor $1+ST_1$ in the denominator where $T_1 = \frac{1}{\omega_{c_1}}$.
On the other hand, if the change in slope at $\omega = \omega_{c_2}$ is $20db/dec$ then the Transfer function has a first order factor $1+ST_2$ in the numerator where $T_2 = \frac{1}{\omega_{c_2}}$.

3) If the change in slope is $-40\, db/dec$, then a doubt arises whether the factor in the denominator is a second order factor or the factor is a multiple pole of the form $(1+TS)^2$.

If the error between asymptotic curve and actual curve is about $-6db$ then the factor of the form $(1+ST_3)^2$ is present in the denominator and if the error is +ve then a quadratic factor of the form $T^2S^2 + 2\xi TS + 1$ is present in the denominator.

4) The value of gain K can be calculated as shown below.
For type-0 system :

1 If the low frequency asymptot has a slope with a horizontal line through A db, then the value of K is

$$K = 10^{A/20}$$

For type-1 system :- (i) If the low frequency asymptot has a slope of -20 db/dec, then extend the line until it intersects the 0 db line. The value of K is equal to the frequency at the point of intersection of the slope with 0 db line.

(or)

(ii) Extend the low frequency asymptot until it intersects $\omega = 1$ frequency line. Find the magnitude A at this point.

$$\therefore \; 20 \log K = A$$

$$K = 10^{A/20}.$$

(or)

(iii) select a point on the low frequency slope, let the magnitude A and frequency ω_1 at that point. Then

$$K = \omega_1 10^{A/20}$$

For type-2 system :- (i) If the low frequency asymptot has a slope of -40 db/dec, then extend the line until it intersects 0 db line. The value of K is equal to the square of the frequency at that point of intersection of the slope with 0 db line.

(or)

(ii) Extend the low frequency asymptot until it intersects $\omega = 1$ frequency line. Find the magnitude A at this point.

$$\therefore \; 20 \log K = A$$

$$K = 10^{A/20}.$$

(or)

(iii) Select a point on the initial slope, let the magnitude A and frequency ω_1 at that point

Thus $20\log\left(\frac{K}{\omega_1^2}\right) = A$

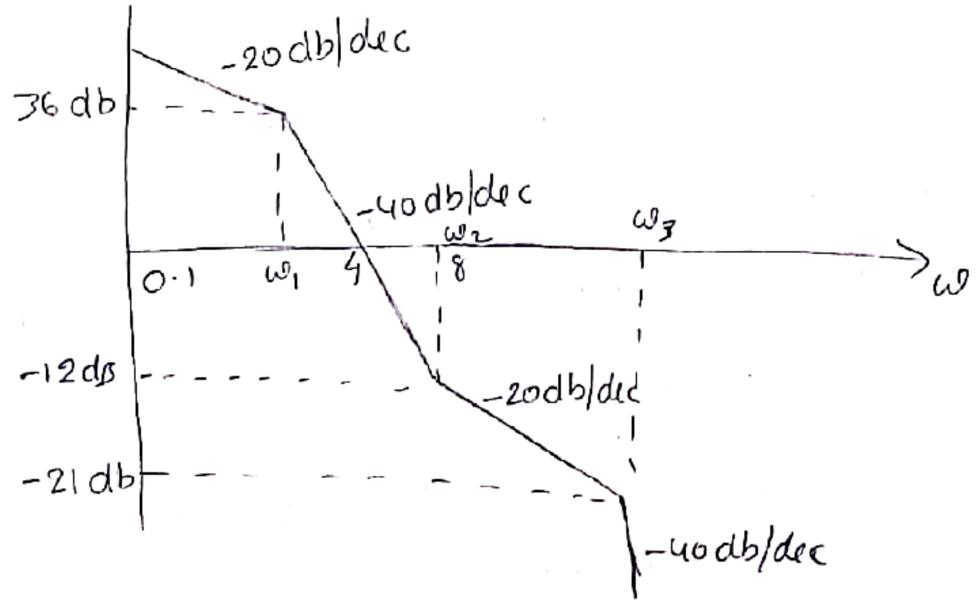
$$K = \omega_1^2\, 10^{A/20}.$$

If the slope of the low frequency asymptote is +ve, then the factors are numerator factors.

The transfer function for individual factors are assembled to get the overall T.F.

PROBLEMS

1) For the Bode plot shown in the fig., Find the transfer function.



SOL:

In the low frequency range, there is an asymptote with slope -20 db/dec. It indicates the system is type-1 and the presence of a factor of the form $\frac{K}{s}$.

At $\omega = \omega_1$, the slope changes to -40 db/dec, i.e. a decrease of -20 db/dec. Thus there exist a factor $1+sT_1$ in the denominator where $T_1 = \frac{1}{\omega_1}$

At $\omega = 4$, the magnitude is zero.

The line joining the magnitudes at $\omega = \omega_1$ & $\omega = 4$ is having a slope of -40 db/dec. The change in magnitude at $\omega = \omega_1$ to $\omega = 4$ is $+36$ db

i.e $$-40 \log 4 - (-40 \log \omega_1) = +36$$

$$+40 \log \frac{4}{\omega_1} = +36$$

corner frequency $\omega_1 = 0.5$

$$\therefore \quad T_1 = \frac{1}{\omega_1} = \frac{1}{0.5} = 2$$

The factor that contributes a -ve slope -20 db/dec at $\omega = \omega_1$ is $\dfrac{1}{1+2s}$

To find K :- At $\omega = \omega_1 = 0.5$, the magnitude is 36 db.

$$\therefore \quad K = \omega_1 \, 10^{A/20}$$

$$= 0.5 \, 10^{36/20} = 31.55$$

At the corner frequency $\omega_2 = 8$, the slope changes from -40 db/dec to -20 db/dec. It indicates the presence of a first order factor $(1 + sT_2)$ in the numerator where $T_2 = \frac{1}{\omega_2} = \frac{1}{8} = 0.125$.

$\therefore$ The first order numerator factor is $(1 + 0.125s)$.

At the corner frequency ω_3, the slope changes from -20 db/dec to -40 db/dec., it indicates the presence of first order factor $(1+T_3 s)$ in the denominator where $T_3 = \frac{1}{\omega_3}$, but ω_3 is not given. The magnitude at ω_3 is -21 db

The frequency ω_8 is 8 rad/sec but the magnitude is (not given) -12db

At $\omega = 4$, the magnitude is 0db and the slope of line is -40db/dec i.e -12 db/octave)

$\therefore$ The change in magnitude is $= -9$ db

Thus $\quad -20\log\omega_3 - (-20\log\omega_2) = -9$

$$20\log \omega_3/\omega_2 = 9$$

$$\omega_3 = 22.55$$

$$\therefore T_3 = \frac{1}{\omega_3} = 0.04435$$

$\therefore$ The first order denominator factor is $\dfrac{1}{1+0.04435\,s}$

$\therefore$ The overall T.F. is $G(s) = \dfrac{31.55\,(1+0.125\,s)}{s\,(1+2s)\,(1+0.04435\,s)}$

2) Sketch the Bode plot for

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$

Determine the GM and PM of the system.

SOL:

Given $\quad G(S) = \dfrac{10}{S(1+0.5S)(1+0.1S)}$

The given T.F is in Time constant form.

Put $\quad s = j\omega$

The sinusoidal T.F., $G(j\omega) = \dfrac{10}{(j\omega)\,(1+j0.5\omega)\,(1+j(0.1\omega))}$

Magnitude PLOT :-

The corner frequencies are

$\omega_{c_1} = \dfrac{1}{0.5} = 2$ rad/sec $\qquad \omega_{c_2} = \dfrac{1}{0.1} = 10$ rad/sec

The various terms of $G(j\omega)$ in increasing order of their corner frequencies are listed in the following table.

Term	corner Frequency (rad/sec)	Slope (db/dec)	change in slope (db/dec)
$\frac{10}{j\omega}$	—	-20	—
$\frac{1}{1+j0.5\omega}$	$\omega_{c_1} = \frac{1}{0.5} = 2$ rad/sec	-20	$-20-20 = -40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c_2} = \frac{1}{0.1} = 10$ rad/sec	-20	$-40-20 = -60$

choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ &

a high " ω_h " $\omega_h > \omega_{c_2}$

let $\omega_l = 0.5$ rad/sec

$\omega_h = 20$ rad/sec

consider $A = 20 \log|G(j\omega)|$ in db

At $\omega = \omega_l = 0.5$ rad/sec, $A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\left(\frac{10}{0.5}\right) = 26$ db

At $\omega = \omega_{c_1} = 2$ rad/sec, $A = 20\log\left(\frac{10}{\omega}\right) = 20\log\left(\frac{10}{2}\right) = 14$ db

At $\omega = \omega_{c_2} = 10$ rad/sec, $A = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right] + A \text{ at } \omega = \omega_{c_1}$

$= -40 \times \log\frac{10}{2} + 14$ db

$= -14$ db

At $\omega = \omega_h = 20$ rad/sec, $A = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c_2}}\right] + A \text{ at } \omega = \omega_{c_2}$

$= -60 \times \log\frac{20}{10} - 14$

$= -32$ db

The magnitude plot can be drawn on the semi log graph sheet and is shown on graph.

Phase Plot :-

The phase angle of $G(j\omega)$ is given by

$$\phi = \angle G(j\omega) = -90 - \tan^{-1}(0.5\omega) - \tan^{-1}(0.1\omega)$$

For different values of ω, the phase angle ϕ can be calculated and are listed in the following table.

ω (rad/sec)	0.5	1	2	5	10	15	20
φ (deg)	−107°	−122°	−146°	−185°	−214°	−229°	−238°

The bode plot of given Transfer function is shown on the semilog graph sheet.

Calculation of GM & PM

From the graph,

gain cross over frequency, $\omega_{gc} = 4.4$ rad/sec

phase cross over frequency, $\omega_{pc} = 4.5$ rad/sec.

∴ Gain Margin, $K_g = -20\log\left|G(j\omega_{pc})\right|$

$$= -20\log\left|\frac{10}{\omega\sqrt{1+(0.5\omega)^2}\sqrt{1+(0.1\omega)^2}}\right|_{\omega=\omega_{pc}}$$

$$= -20\log\left(\frac{10}{4.5\sqrt{1+(0.5\times4.5)^2}\sqrt{1+(0.1\times4.5)^2}}\right)$$

$$= -(-1.69)\ \text{db}$$

$$GM = 1.69\ \text{db}$$

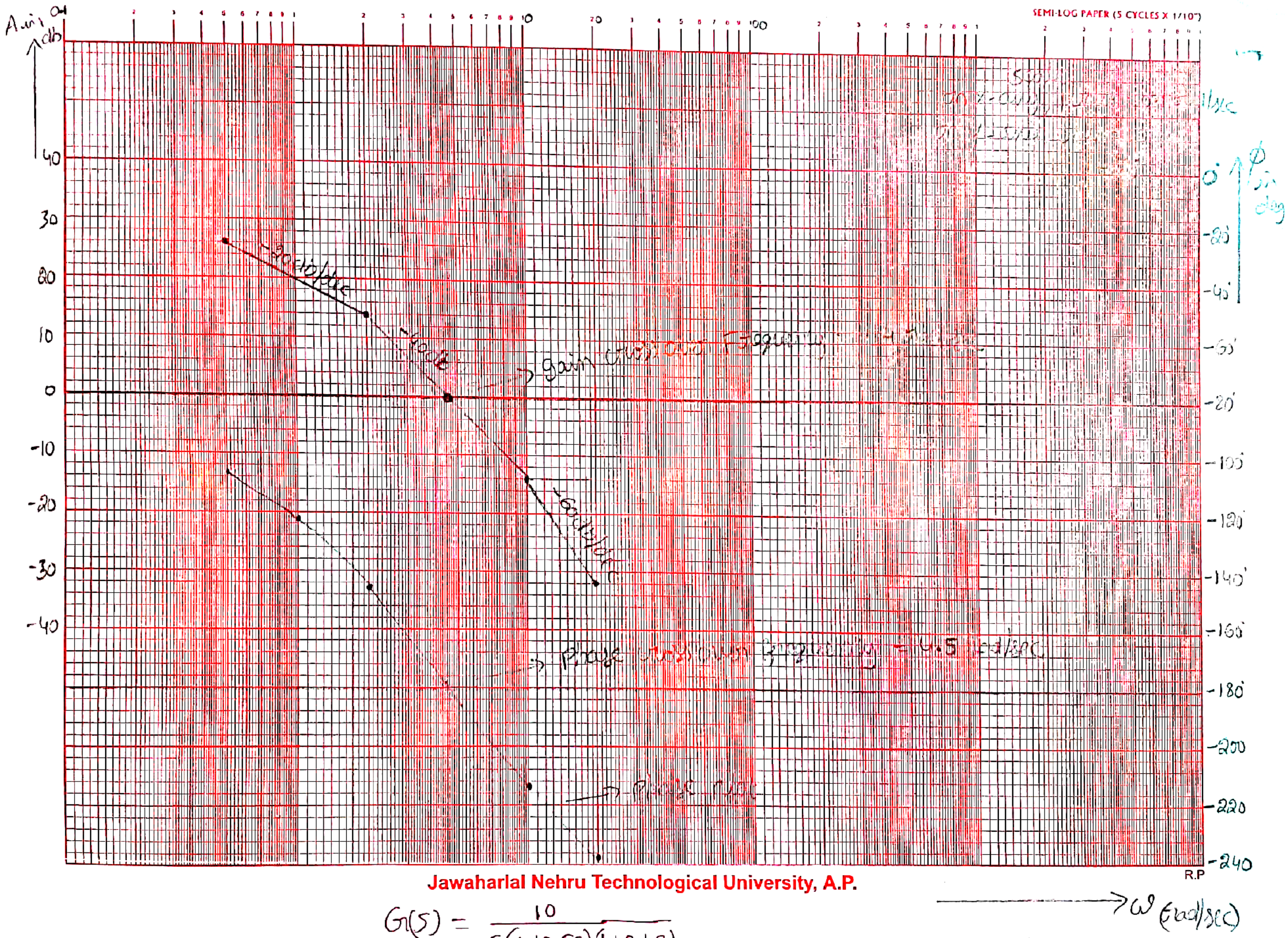
∴ phase Margin, $\gamma = 180 + \phi_{gc}$

$$= 180 - 90 - \tan^{-1}(0.5\,\omega_{gc}) - \tan^{-1}(0.1\,\omega_{gc})$$

$$= 180 - 90 - \tan^{-1}(0.5\times4.4) - \tan^{-1}(0.1\times4.4)$$

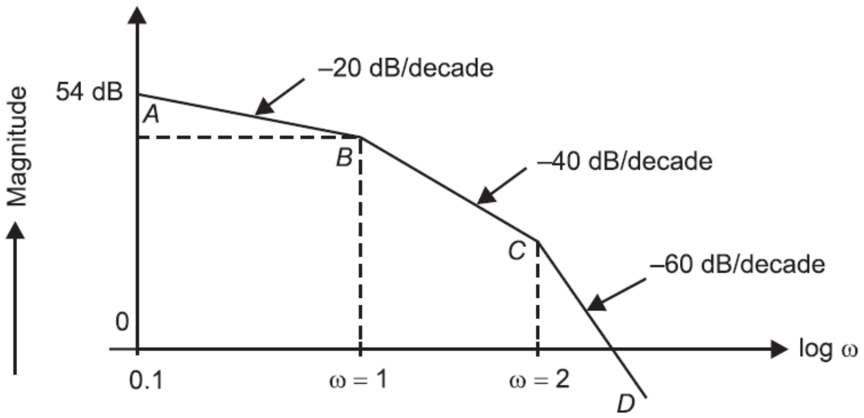
PM, $\gamma = 0.69°$

Both GM & PM are +ve, then the system is stable.



$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$

3) Find the open-loop transfer function of a system whose approximate Bode plot is shown in Figure below.



SOL:

The initial slope of the Bode plot is – 20 dB/decade.

For a slope of –20 dB/decade, the system is type 1. The system transfer function is expressed as

$$G(s)\,H(s) = \frac{K}{s}$$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega}$$

$$20\log_{10}\left|G(j\omega)H(j\omega)\right| = 20\log_{10}K - 20\log_{10}\omega$$

Substituting values,

$$54 = 20\log_{10}K - 20\log_{10}(0.1)$$

$$20\log_{10}K = 54 + 20\times(-1)$$

$$20\log_{10}K = 54 - 20 = 34$$

$$\log_{10}K = 1.57$$

$$K = 52.2$$

Thus, for the initial part of the Bode plot, we get the transfer function

$$TF = \frac{K}{s} = \frac{52.2}{s}$$

At corner frequency, $\omega = 1$ and the slope has changed by another –20 dB/decade. The slope is negative. The corresponding factor of the TF is $1/(1 + s)$.

At the corner frequency $\omega = 2$, the slope is increased by another –20 dB/decade. The slope is negative. Hence the corresponding factor of the TF is $1/(1 + 0.5s)$.

Thus, the transfer function of the control system is

$$G(s)H(s) = \frac{K}{s(1+s)(1+0.5s)} = \frac{52.2}{s(1+s)(1+0.5s)}$$

4) Draw the Bode plot for a control system having transfer function,

$$G(s)H(s) = \frac{100}{s(s+1)(s+2)}$$

Determine the following from the Bode plot:
(1) Gain margin; (2) Phase margin; (3) Gain crossover frequency and
(4) Phase crossover.
SOL:

Let us substitute $s = j\omega$ in the transfer function as

$$G(j\omega)H(j\omega) = \frac{100}{j\omega(1+j\omega)(2+j\omega)}$$

$$= \frac{50}{j\omega(1+j\omega)(1+j0.5\omega)}$$

(1) Corner frequencies are $\omega_1 = 1$ rad/sec

$$\omega_2 = \frac{1}{0.5} = 2 \text{ rad/sec}$$

(2) The starting of the Bode plot is taken as lower than the lowest frequency. Since lowest corner frequency here is 1 rad/sec, we can take starting frequency as 0.1 rad/sec.

(3) By examining the transfer function we see that it represents a type 1 system (power of s in the denominator is 1). So the initial slope is –20 dB/decade and continues to corner frequency, $\omega = 1$ rad/sec.

(4) Corner frequencies, $\omega = 1$ rad/sec is due to term $1/(1+j\omega)$ of the TF. Therefore, the Bode plot after this frequency will have a further slope of –20 dB/decade. Thus, the total slope will become –40 dB/dec. This slope will continue till the next corner frequency, $\omega_2 = 2$ rad/sec. This corner frequency of ω_2 is due to the term $1/(1+j0.5\omega)$ of the TF; due to which there will be another increase of –20 dB in the slope of the Bode plot at $\omega = 2$ rad/sec. Thus, the total slope at frequencies higher than $\omega = 2$ rad/sec will be –60 dB/decade.

(5) The phase angle $\phi = \angle G(j\omega)H(j\omega)$ for a range of frequencies is calculated as follows.

$$G(j\omega)H(j\omega) = \frac{50}{(0+j\omega)(1+j\omega)(1+0.5j\omega)}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}\frac{\omega}{0} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{0.5\omega}{1}$$

or

$$\phi = -90^\circ - \tan^{-1}\omega - \tan^{-1}0.5\omega$$

Phase angle ϕ at different values of ω have been calculated as following

ω	0	0.1	0.2	0.5	1.0	1.3	1.5	2	4.5
ϕ	–90°	–98.6°	107°	–130°	–161.6°	–175.5°	183.2°	198.4°	233°

The magnitudes in dB at different frequencies, that is, at initial and corner frequencies are calculated as follows:

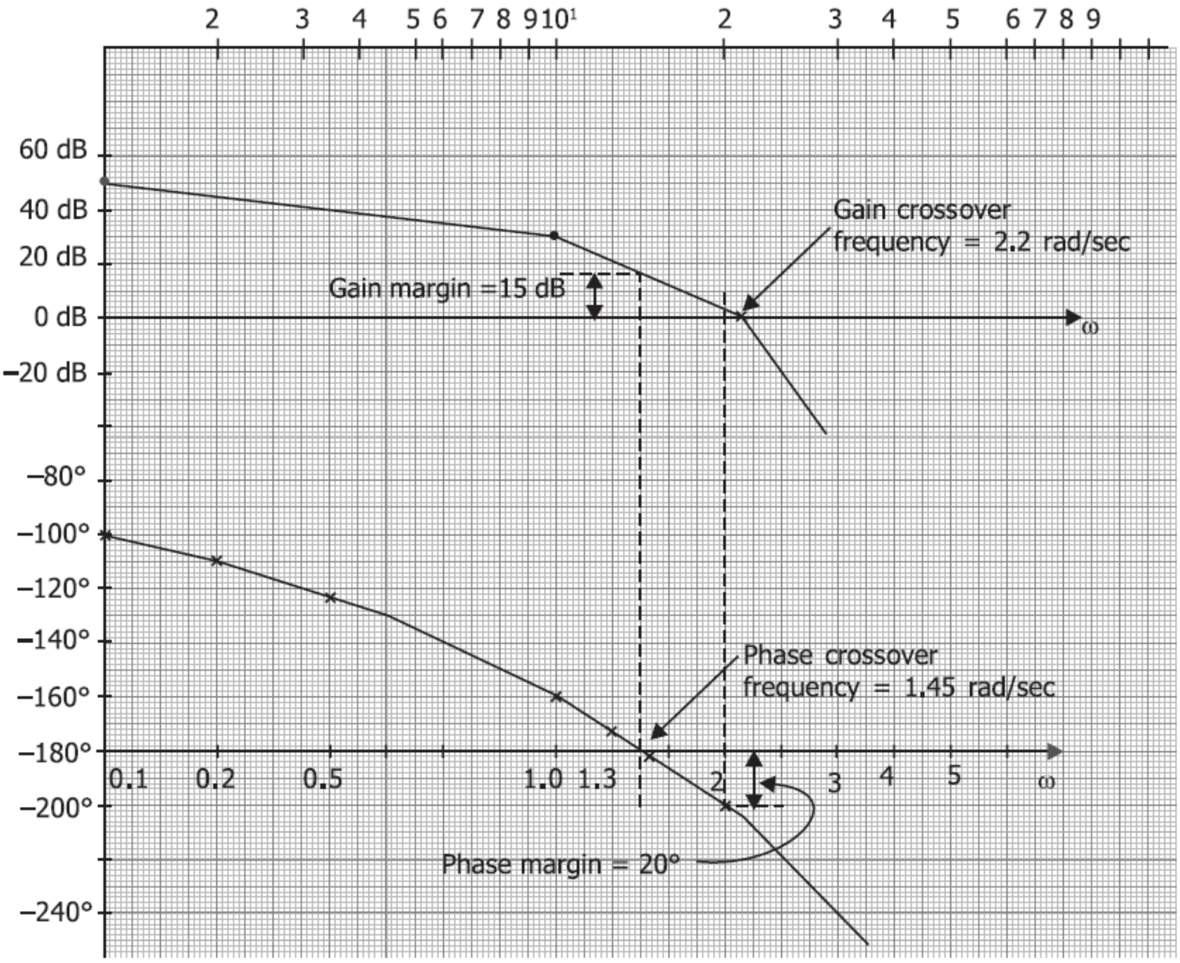
$\omega = 0.1$	Magnitude $\left\lvert \dfrac{50}{j\omega} \right\rvert$	$= 20\log 50 - 20\log\omega$ $= 20\log 50 - 20\log(0.1)$ $= 54$ dB
$\omega = 1.0$	Magnitude $\left\lvert \dfrac{50}{j\omega(1+j\omega)} \right\rvert$	$= 20\log 50 - 20\log\omega - 20\log\sqrt{1+\omega^2}$ $= 20\log 50 - 20\log 1 - 20\log\sqrt{2}$ $= 30$ dB
$\omega = 2.0$	Magnitude $\left\lvert \dfrac{50}{j\omega(1+j\omega)(1+j0.5\omega)} \right\rvert$	$= 20\log 50 - 20\log 2 - 20\log\sqrt{1^2+2^2}$ $-20\log\sqrt{1^2+(.5\times 2)}$ $= 5$ dB

Figure shows the Bode plot for magnitude and phase angle drawn on log scale. The gain margin is calculated at the phase crossover frequency and phase margin is calculated at gain crossover frequency. The results as found are

Gain margin = 15 dB; Phase margin = 20°

Gain crossover frequency = 2.2 rad/sec;

Phase crossover frequency = 1.45 rad/sec.



5) Draw the Bode plot for a control system having transfer function,

$$G(s)H(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$

Determine the Gain margin & Phase margin from the Bode plot. State the system is stable or not.

SOL:

$$G(s)H(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$

$$= \frac{10(0.1s+1)}{s(0.5s+1)(0.2s+1)} \quad \text{[dividing numerator and denominator by 10]}$$

(1) The corner frequencies are

$$\omega_1 = \frac{1}{0.5} = 2 \text{ rad/sec}$$

$$\omega_2 = \frac{1}{0.2} = 5 \text{ rad/sec}$$

$$\omega_3 = \frac{1}{0.1} = 10 \text{ rad/sec}$$

(2) Starting frequency of the Bode plot is taken as lower than the lowest corner frequency. Here the lowest corner frequency is 2 rad/sec. We can take starting frequency of say, 1 rad/sec.

(3) The system is a type 1 system since the power of s in the denominator of the transfer function is 1. So, the initial slope will be –20 dB/decade. This slope will continue till the corner frequency of $\omega = 2$ rad/sec is reached.

(4) The corner frequency of $\omega = 2$ rad/sec is due to the term $1/(0.5s + 1)$ for which $T(j\omega) = 1/[1 + j\omega(0.5)]$. The slope of the Bode plot after this frequency will change by –20 dB/decade. Thus, the Bode plot after $\omega = 2$ rad/sec will have a slope of –40 dB/decade and continue till the next corner frequency of $\omega = 5$ rad/sec is reached. Corner frequency, $\omega = 5$ rad/sec is due to the term $1/(0.2s + 1)$ of the transfer function for which we can write $T(j\omega) = 1/[1 + j\omega(0.2)]$. The slope of the Bode plot after $\omega = 5$ rad/sec will change by another –20 dB/decade, making the total slope equal to –60 dB/decade. This slope will continue till the next corner frequency of $\omega = 10$ rad/sec is reached.

The corner frequency of $\omega = 10$ rad/sec is due to the term $(1 + 0.1s)$ appearing at the numerator of the transfer function which can be written as $T(j\omega) = 1/[1 + j\omega(0.1)]$. The Bode plot after this corner frequency will change by +20 dB/decade. The slope of the Bode plot after $\omega = 10$ rad/sec will therefore be –40 dB/dec and will continue for higher frequencies.

5. Now we will calculate the phase angle $\phi(\omega)$ for the transfer function.

$$G(j\omega)\,H(j\omega) = \frac{10(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.2\omega)}$$

$$\phi(\omega) = \underline{|G(j\omega)H(j\omega)} = -90^\circ - \tan^{-1}\frac{0.5\omega}{1} - \tan^{-1}\frac{0.2\omega}{1} + \tan^{-1}\frac{0.1\omega}{1}$$

The values of ϕ at different values of ω are calculated as follows.

ω rad/sec	0	0.1	0.5	1	2	5	8	10	20
ϕ	–90°	–93°	–107°	–122°	–145°	–176°	–185.3°	–187°	–205°

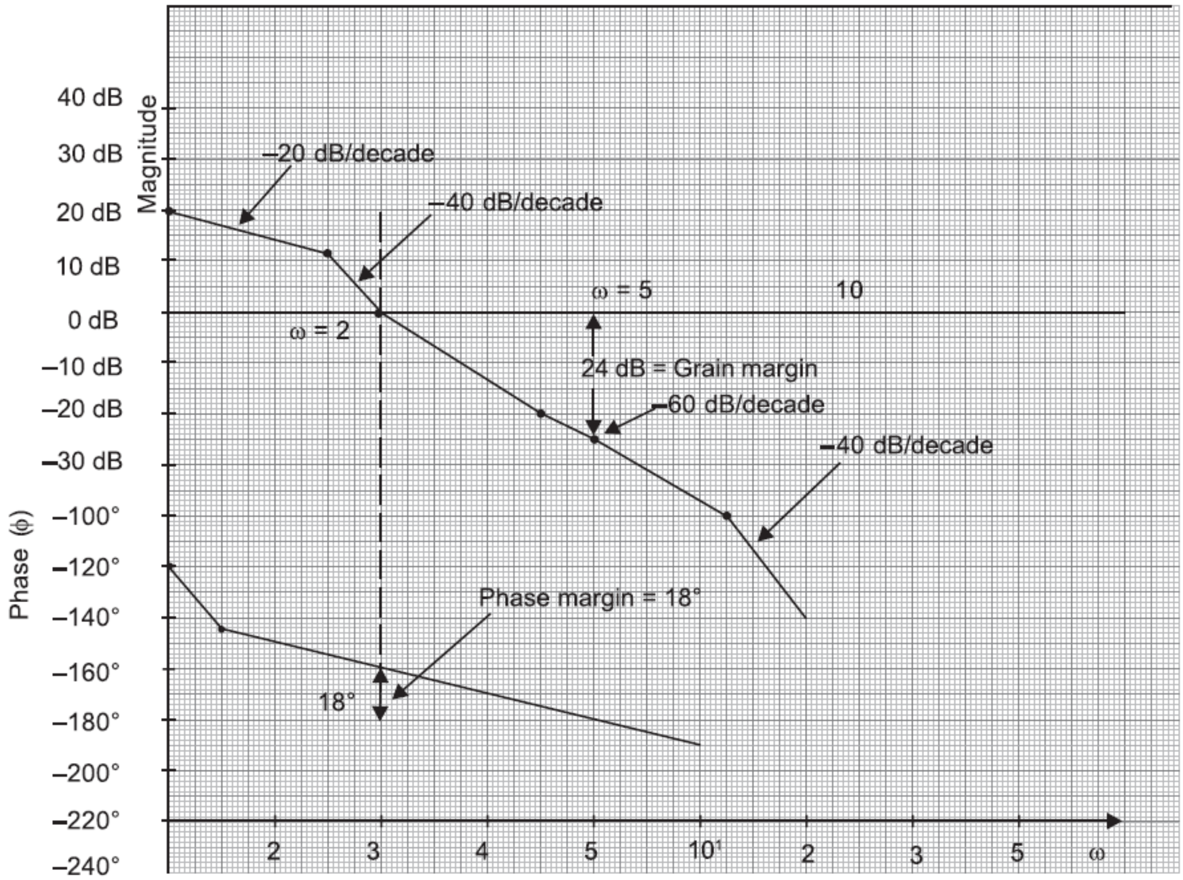
The magnitudes are calculated as follows:

Frequency	Magnitude
$\omega = 1$ rad/sec	$\left\lvert\frac{K}{j\omega}\right\rvert = 20\log K - 20\log\omega$ $= 20\log_{10} - 20\log(1)$ $= 20$ dB
$\omega = 2$ rad/sec	$\left\lvert\frac{K}{j\omega(1 + j\omega 0.5)}\right\rvert = 20\log k - 20\log\omega - 20\log\sqrt{1^2 + .25\omega^2}$ $= 20\log 10 - 20\log 2 - 20\log\sqrt{1^2 + .25\times 2^2}$ $= 11$ dB
$\omega = 5$ rad/sec	$\left\lvert\frac{K}{j\omega(1 + j\omega 0.5)(1 + j\omega 0.2)}\right\rvert$ $= 20\log K - 20\log\omega - 20\log\sqrt{1^2 + .25\omega^2} - 20\log\sqrt{1^2 + 0.4\omega^2}$ $= 20\log 10 - 20\log 5 - 20\log\sqrt{1^2 + .25\times 25} - 20\log\sqrt{1^2 + .04\times 25}$ $= -20$ dB
$\omega = 10$ rad/sec	$\left\lvert\frac{10(1 + j\omega 0.1)}{j\omega(1 + j\omega 0.5)(1 + j\omega 0.2)}\right\rvert$ $= 20\log 10 - 20\log 10 - 20\log\sqrt{1+25} - 20\log\sqrt{1+4} + 20\log\sqrt{1+1}$ $= -31$ dB

The Bode plot has been drawn using the above data (Figure). The phase margin at gain crossover frequency = 18°. The gain margin calculated at phase crossover frequency = 24 dB

GM = Initial value – Final value = (0) dB – (–24 dB) = 24 dB

Since both phase margin and gain margin are positive, the system is stable.



6) Sketch the bode plot for the following transfer function and determine gain cross over frequency and phase cross over frequency

$$G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$$

SOL:

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in G(s).

$$\therefore G(j\omega) = \frac{5\,(1+j2\omega)}{(1+j4\omega)\,(1+j0.25\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{4} = 0.25$ rad / sec, $\omega_{c2} = \frac{1}{2} = 0.5$ rad / sec, $\omega_{c3} = \frac{1}{0.25} = 4$ rad / sec

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by the each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$. Let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec.

Let A = |G(jω)| in db and let us calculate A at ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/deg
5	–	0	–
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	–20	0 – 20 = –20
1 + j2ω	$\omega_{c2} = \frac{1}{2} = 0.5$	20	–20 + 20 = 0
$\frac{1}{1+j0.25\omega}$	$\omega_{c3} = \frac{1}{0.25} = 4$	–20	0–20 = –20

At $\omega = \omega_l$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

At $\omega = \omega_{c1}$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -20 \times \log\frac{0.5}{0.25} + 14 = +8\text{db}$$

$$\text{At } \omega = \omega_{c3},\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = 0 \times \log\frac{4}{0.5} + 8 = +8\text{db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c3}}\right] + A_{(\text{at } \omega = \omega_{c3})} = -20 \log\frac{10}{4} + 8 = 0 \text{ db}$$

Let the points a, b, c, d and e be the points correponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales on x-axis. Fix the points a, b, c, d and e on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

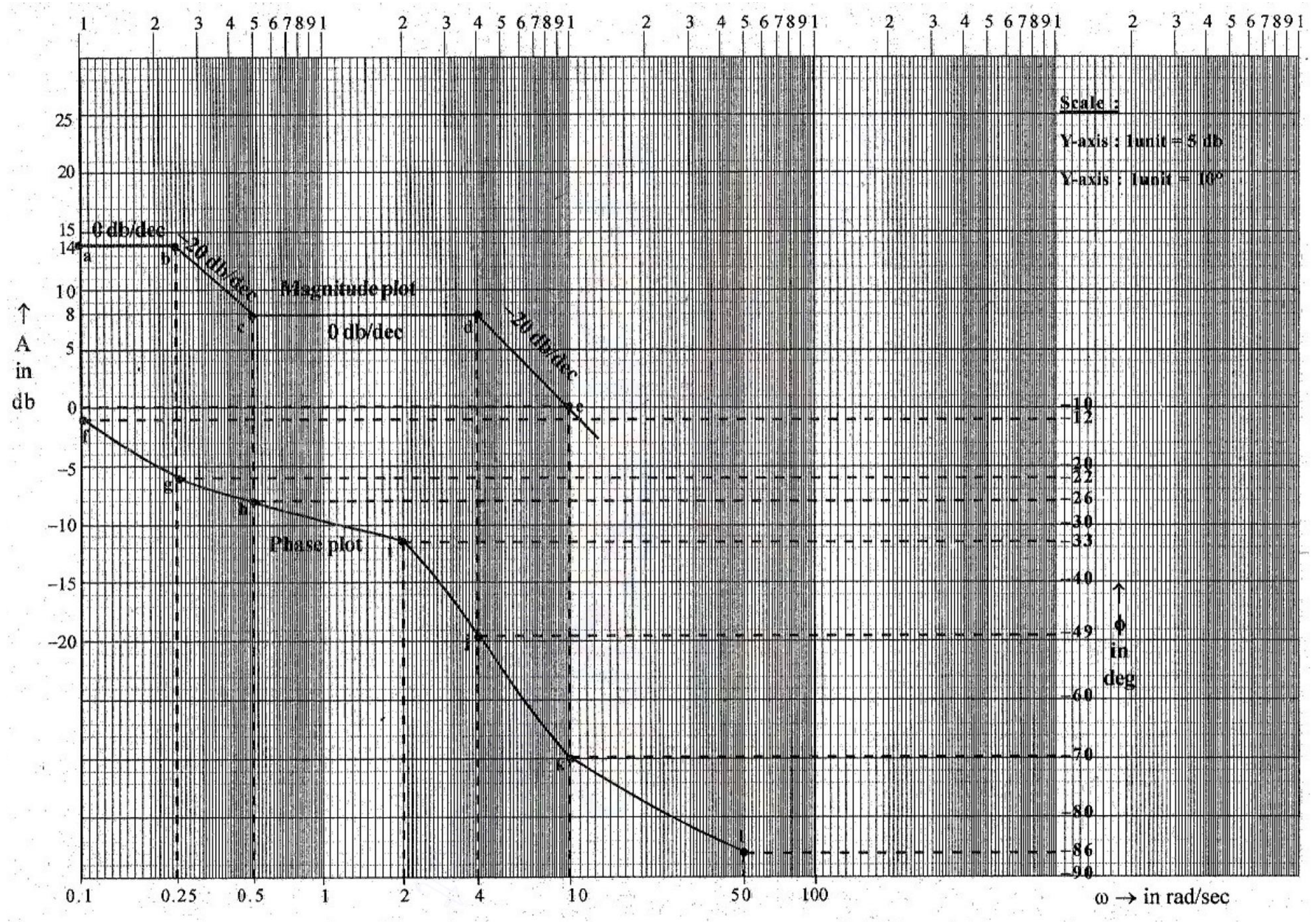
The phase angle of G(jω), $\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$

The phase angle of G(jω) are calculated for various values of ω and listed in the table-2.

TABLE-2

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$	Points in phase plot
0.1	11.3	21.8	1.43	$-11.93 \approx -12$	f
0.25	26.56	45.0	3.5	$-21.94 \approx -22$	g
0.5	45.0	63.43	7.1	$-25.53 \approx -26$	h
2	75.96	82.87	26.56	$-33.47 \approx -33$	i
4	82.87	86.42	45.0	$-48.55 \approx -49$	j
10	87.13	88.56	68.19	$-69.62 \approx -70$	k
50	89.42	89.71	85.42	$-85.71 \approx -86$	l

On the same semilog graph sheet choose a scale of 1unit = 10° on y-axis on the right side of the semilog graph sheet. Mark the calculated phase angle on the graph sheet, Join the points by a smooth curve. The magnitude and phase plots are shown in fig.


Scale :
Y-axis : 1unit = 5 db
Y-axis : 1unit = 10°
Magnitude plot
0 db/dec
-20 db/dec
0 db/dec
-20 db/dec
Phase plot
A in db
φ in deg
ω → in rad/sec

7) Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s)=\frac{75(1+0.2s)}{s(s^2+16s+100)}$$

SOL:

On comparing the quadratic factor in the denominator of G(s) with standard form of quadratic factor we can estimate ζ and ω_n.

$$\therefore s^2+16s+100 = s^2+2\zeta\omega_n s+\omega_n^2$$

On comparing we get,

$$\omega_n^2=100 \quad \Rightarrow \quad \omega_n=10$$

$$2\zeta\omega_n=16 \quad \Rightarrow \quad \zeta=\frac{16}{2\omega_n}=\frac{16}{2\times 10}=0.8$$

Let us convert the given s-domain transfer function into bode form or time constant form.

$$\therefore G(s)=\frac{75(1+0.2s)}{s\left(s^2+16s+100\right)}=\frac{75\,(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)}=\frac{0.75\,(1+0.2s)}{s\left(1+0.01s^2+0.16s\right)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in G(s).

$$\therefore\ G(j\omega)=\frac{0.75\,(1+0.2\,j\omega)}{j\omega\left(1+0.01(j\omega)^2+0.16j\omega\right)}=\frac{0.75\,(1+j0.2\omega)}{j\omega\left(1-0.01\omega^2+j0.16\omega\right)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1}=\frac{1}{0.2}=5$ rad / sec and $\omega_{c2}=\omega_n=10$ rad / sec

Note : For the quadratic factor the corner frequency is ω_n.

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	-	-20	
$1+j0.2\omega$	$\omega_{c1}=\frac{1}{0.2}=5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2}=\omega_n=10$	- 40	$0-40=-40$

Choose a low frequency ω_l such that $\omega_l<\omega_{c1}$ and choose a high frequency ω_h such that $\omega_h>\omega_{c2}$.

Let, $\omega_l=0.5$ rad/sec and $\omega_h=20$ rad/sec.

Let, $A=|G(j\omega)|$ in db .

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At,} \quad \omega = \omega_l, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{0.5} = 3.5 \text{ db}$$

$$\text{At,} \quad \omega = \omega_{c1}, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{5} = -16.5 \text{ db}$$

$$\text{At,} \quad \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= 0 \times \log\frac{10}{5} + (-16.5) = -16.5 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -40 \times \log\frac{20}{10} + (-16.5) = -28.5 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a,b,c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

> *Note : In quadratic factors the phase varies from 0° to 180°. But calculator calculates $\tan^{-1}$ only between 0° to 90°. Hence a correction of 180° should be added to phase after ω_n.*

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} \quad \text{for } \omega \le \omega_n$$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \quad \text{for } \omega > \omega_n$$

The phase angle of G(jω) are calculated for various values of w and listed in Table-2.

TABLE-2

ω rad/sec	$\tan^{-1}0.2\omega$ deg	$\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
1	11.3	9.2	$-87.9 \approx -88$	f
5	45	46.8	$-91.8 \approx -92$	g
10	63.4	90	$-116.6 \approx -116$	h
20	75.9	$-46.8+180 = 133.2$	$-147.3 \approx -148$	i
50	84.3	$-18.4+180 = 161.6$	$-167.3 \approx -168$	j
100	87.1	$-92+180 = 170.8$	$-173.7 \approx -174$	k

On the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

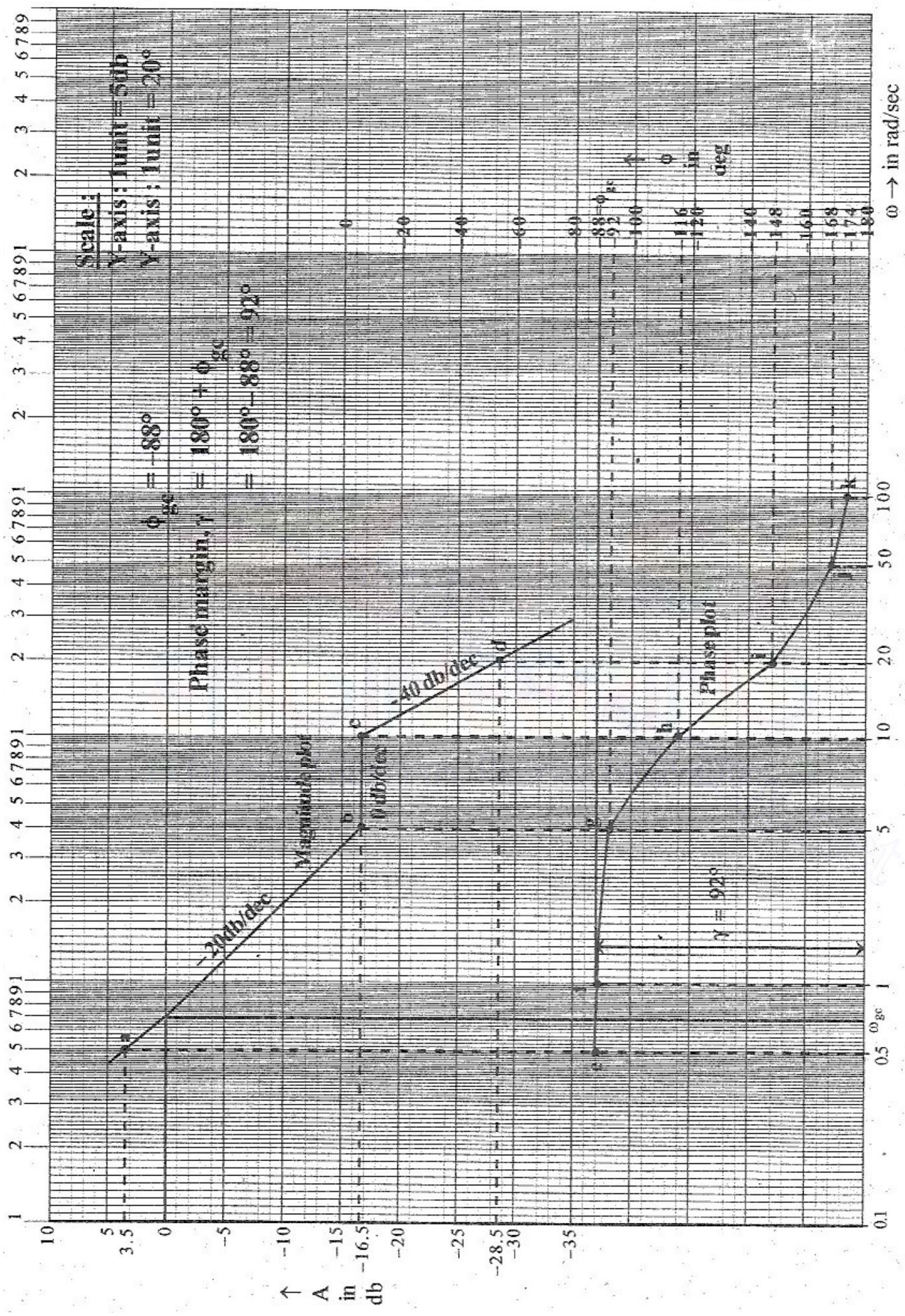
Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross-over frequency, ω_{gc}.

we get, $\phi_{gc} = 88°$

$\therefore$ Phase margin, $g = 180° + \phi_{gc} = 180° - 88° = 92°$

The phase plot crosses $-180°$ only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ db.

Hence gain margin is $+\infty$.



POLAR PLOT

The sinusoidal T.F. $G(j\omega)$ is a complex function

i.e. $G(j\omega) = Re[G(j\omega)] + j\,Im[G(j\omega)]$

$\Rightarrow G(j\omega) = |G(j\omega)| \angle G(j\omega)$

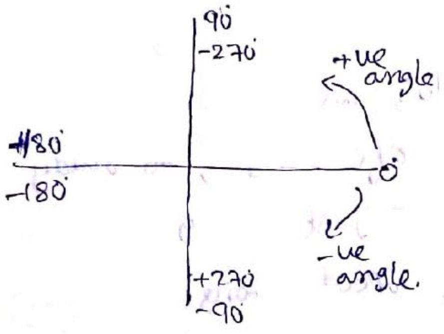
$= M \angle \phi$

As the input frequency ω is varied from $0-\infty$, the magnitude M and phase angle ϕ change and hence the tip of the phasor $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is called "Polar plot". i.e.

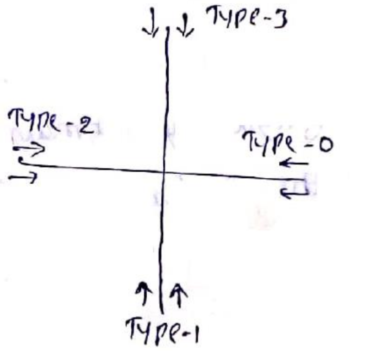
* The polar plot is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar co-ordinates as ω is varied from zero to infinity. Thus polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from $0-\infty$.

To plot the polar plot on ordinary graph sheet compute the magnitude and phase for various values of ω. Then convert the polar coordinates to rectangular coordinates. Sketch the polar plot using rectangular coordinates.

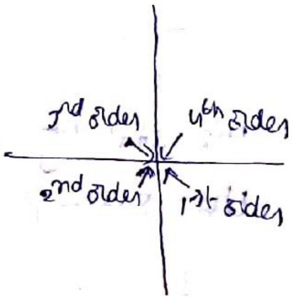
For minimum phase T.F., with only poles, the type number of the system determines at what quadrant the polar plot starts and the order of the system determines at what quadrant the polar plot ends.



Polar Graph



Start of polar plot



End of polar plot

PROCEDURE TO SKETCH POLAR PLOT

1) Determine the T.F G(s) of the system.
2) Put $s = j\omega$ in the T.F. to obtain $G(j\omega)$.
3) At $\omega = 0$ & $\omega = \infty$, calculate $|G(j\omega)|$ $\angle G(j\omega)$.
4) Rationalize the function $G(j\omega)$ and seperate real & imag. terms
5) Equating the Imaginary part to zero and determine the frequencies at which plot intersects the real axis and calculate the value of $G(j\omega)$ at this frequencies.
6) Equating the real part to zero and determine the freq at which the plot intersects the imaginary axis and calculate the value of $G(j\omega)$ at this frequencies.
7) Sketch the polar plot.

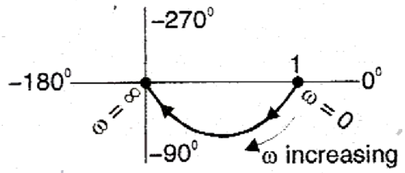
TYPICAL SKETCHES OF POLAR PLOT

Type : 0, Order : 1 $\quad G(s) = \dfrac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\,\angle \tan^{-1}\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}}\angle -\tan^{-1}\omega T$$

As $\omega \to 0$, $\quad G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -90°$

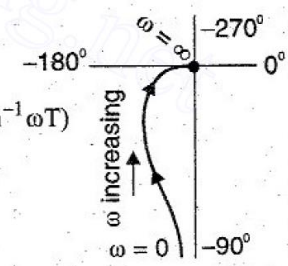


Type : 1, Order : 2 $\quad G(s) = \dfrac{1}{s(1+sT)}$

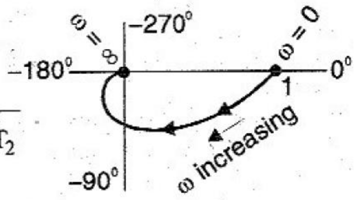
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90°\sqrt{1+\omega^2T^2}\angle\tan^{-1}\omega T} = \frac{1}{\omega\sqrt{1+\omega^2T^2}}\angle(-90° - \tan^{-1}\omega T)$$

As $\omega \to 0$, $\quad G(j\omega) \to \infty\angle -90°$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -180°$



Type : 0, Order : 2 $G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$



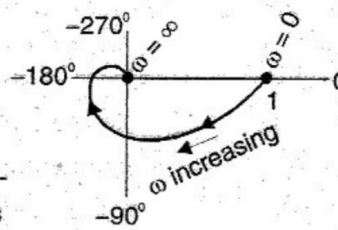
$$G(j\omega) = \frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2}$$

$$= \frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -180^\circ$

Type : 0, Order : 3 $G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$



$$G(j\omega) = \frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

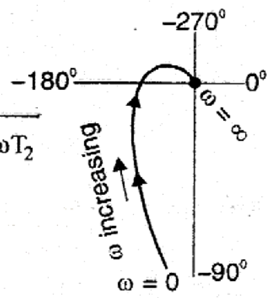
$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

$$= \frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3)$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -270^\circ$

Type : 1, Order : 3 $G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$



$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)\,(1+j\omega T_2)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\ \sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

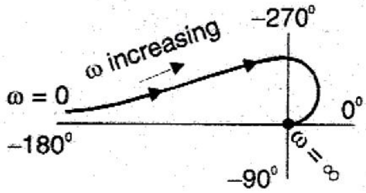
As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -270^\circ$

Type : 2, Order : 4 $G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$

$$G(j\omega) = \frac{1}{(j\omega)^2\ (1+j\omega T_1)\ (1+j\omega T_2)} = \frac{1}{\omega^2\angle -180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

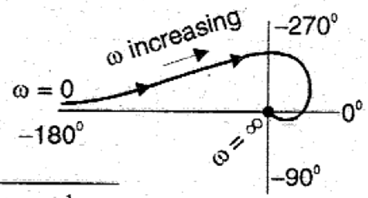
$$= \frac{1}{\omega^2\ \sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$



As $\omega \to 0$, $G(j\omega) \to \infty\angle -180^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -360^\circ$

Type : 2, Order : 5 $G(s)=\dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$



$$G(j\omega)=\frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

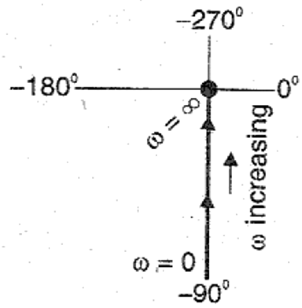
$$=\frac{1}{\omega^2\angle-180^\circ\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

$$=\frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $G(j\omega)\to\infty\angle-180^\circ$

As $\omega\to\infty$, $G(j\omega)\to 0\angle-450^\circ=0\angle-90^\circ$

Type : 1, Order : 1 $G(s)=\dfrac{1}{s}$

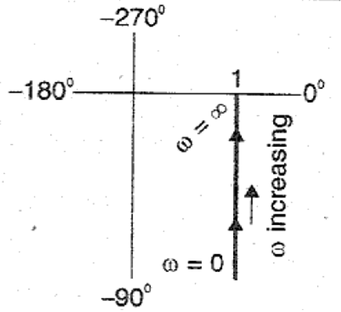


$$G(j\omega)=\frac{1}{j\omega}=\frac{1}{\omega\angle90^\circ}=\frac{1}{\omega}\angle-90^\circ$$

As $\omega\to 0$, $G(j\omega)\to\infty\angle-90^\circ$

As $\omega\to\infty$, $G(j\omega)\to 0\angle-90^\circ$

$$G(s)=\frac{1+sT}{sT}$$

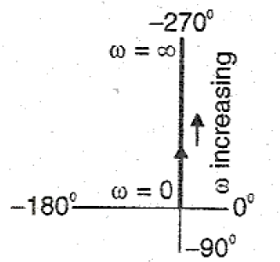


$$G(j\omega)=\frac{1+j\omega T}{j\omega T}=\frac{1}{j\omega T}+1=\frac{1}{\omega T\angle90^\circ}+1=\frac{1}{\omega T}\angle-90^\circ+1$$

As $\omega\to 0$, $G(j\omega)\to\infty\angle-90^\circ+1$

As $\omega\to\infty$, $G(j\omega)\to 0\angle-90^\circ+1$

$G(s)=s$

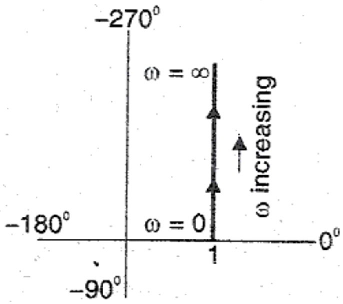


$G(j\omega)=j\omega=\omega\angle90^\circ$

As $\omega\to 0$, $G(j\omega)\to 0\angle90^\circ$

As $\omega\to\infty$, $G(j\omega)\to\infty\angle90^\circ$

$G(s)=1+sT$



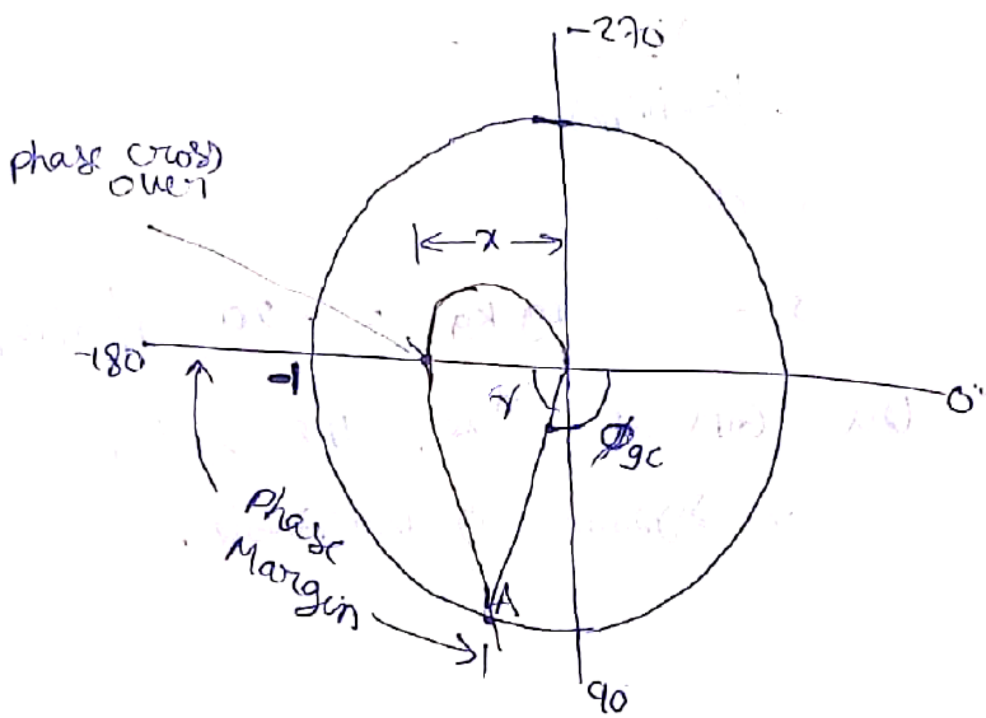
$G(j\omega)=1+j\omega T=1+\omega T\angle90^\circ$

As $\omega\to 0$, $G(j\omega)\to 1+0\angle90^\circ$

As $\omega\to\infty$, $G(j\omega)\to 1+\infty\angle90^\circ$

DETERMINATION OF PHASE MARGIN(PM), GAIN MARGIN(GM) AND STABILITY FROM POLAR PLOT

Consider the Polar Graph as shown below.



Phase Margin :- (PM)

The phase Margin is that amount of additional phase lag at the gain crossover freq. required to bring the system to the verge of instability.

A circle with radius equal to unity is drawn and it intersects the polar plot at point A where the freq is ω_{gc}. i.e. gain cross over freq.. The gain cross over freq. is the freq at which the magnitude is unity $|G(j\omega)| = 1$.

The phase margin is given by

$$\gamma = 180° + \phi_{gc}$$

If the phase margin is +ve, the System is stable. The Point $-1+j0$ is critical point. when the cross over point to the left of $-1+j0$, the phase margin is -ve, the system is unstable.

Gain Margin (GM)

The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the phase crossover freq. i.e. the phase at $-180°$

$$GM, \ K_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{(x)}$$

But GM is in decibles.

$$K_g \ dB = 20 \log K_g = -20 \log|G(j\omega_{pc})|$$

when the GM is +ve, the system is stable. If GM is -ve, the system is unstable.

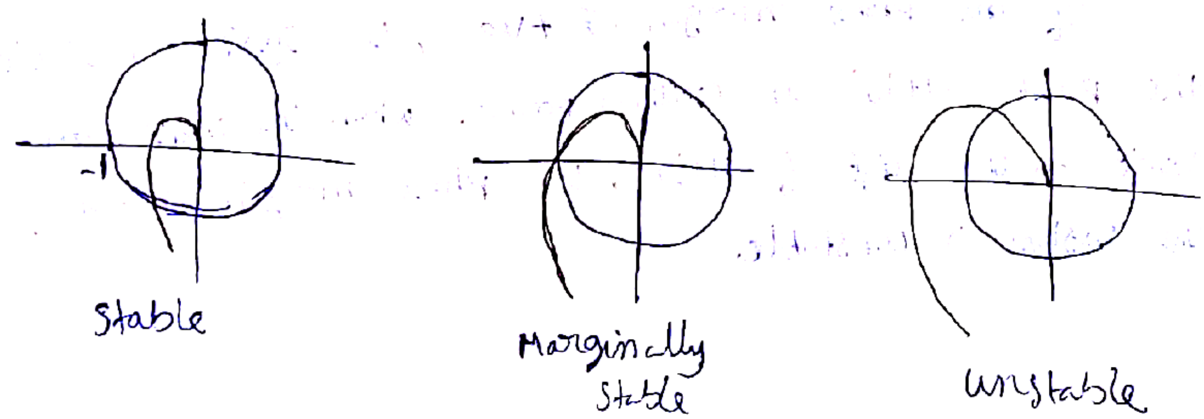
Stability

The Polar plot can also be used to determine the stability of the system.

If the polar plot is with in the circle i.e. the critical point $-1+j0$ is outside the polar plot, the system is called stable system.

If the polar plot is passes through the critical point $-1+j0$, the system is called marginally stable. system.

If the polar plot is out side the circle i.e. the critical point $-1+j0$ is with in the polar plot, the system is called unstable system.



PROBLEMS

1) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)(1+2s)}$. Sketch the polar plot and determine gain margin and phase margin.

SOL:

Put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega)(1+j2\omega)}$$

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}}$$

$$\phi = \angle G(j\omega) = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

The corner frequencies are 0.5 rad/sec & 1 rad/sec

From Graph,

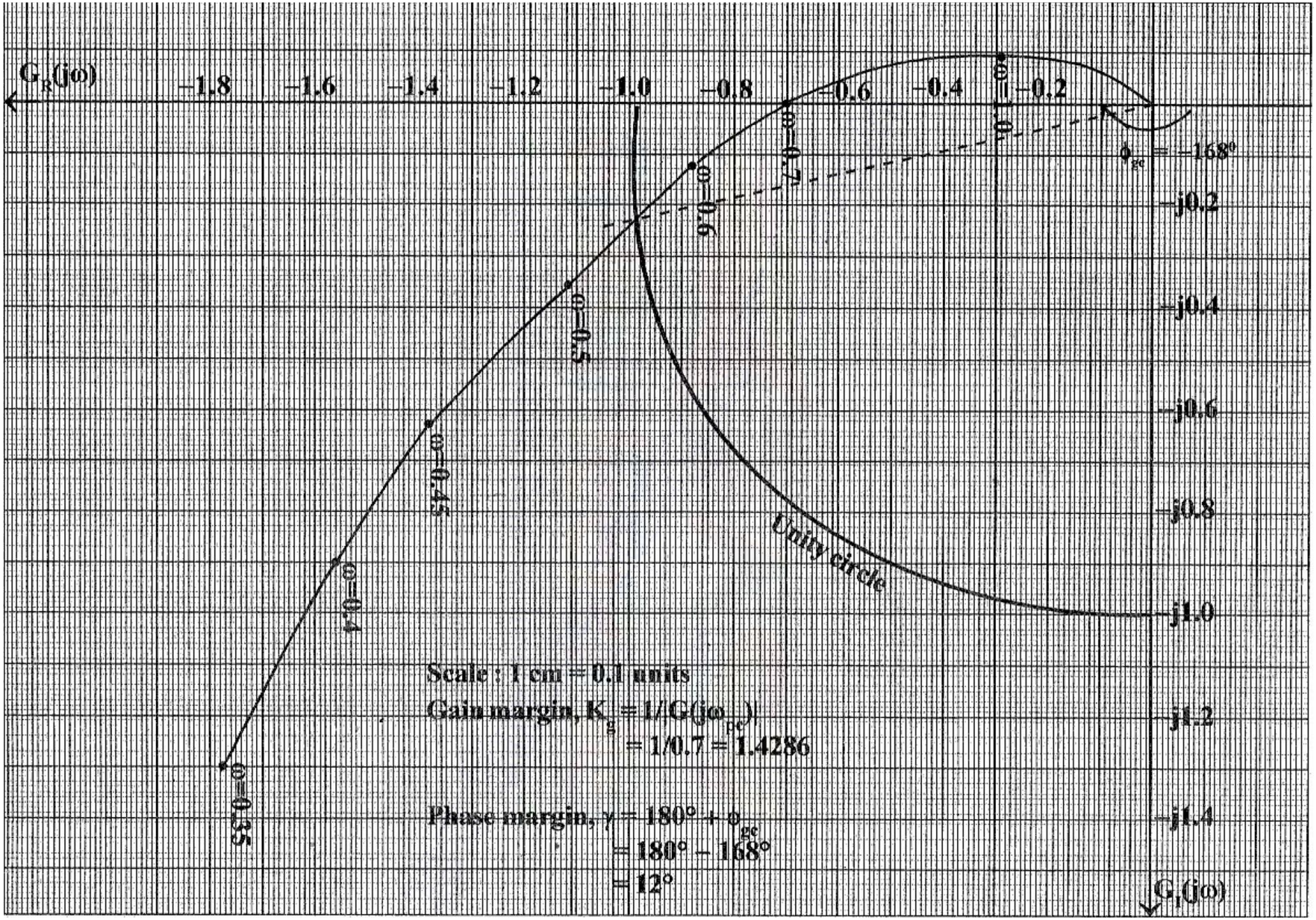
The gain Margin $K_g = \frac{1}{0.7} = 1.428$

$$GM \text{ in } dB = +20\log(1.428)$$
$$= +3.09 \text{ dB}$$

The phase Margin, $\gamma = 180 - 168^\circ$
$$= \underline{12^\circ}$$

For different values of ω, the magnitude and the phase angle are tabulated below.

ω	M	ϕ	M cosϕ + jM sinϕ
0.5	1.26	−161.56	−1.19 − j0.398
0.55	1.07	−166.5	−1.04 − j0.249
0.6	0.91	−171.16	−0.899 − j0.139
0.65	0.786	−175.45	−0.77 − j0.06
0.7	0.68	−179.45	−0.68 − j0.006
0.75	0.59	−183.18	−0.59 + j0.03
0.8	0.517	−186.65	−0.5 + j0.05
0.85	0.45	−189.89	−0.44 + j0.07
0.9	0.4	−192.93	−0.39 + j0.09
0.95	0.355	−195.77	−0.37 + j0.095
1	0.316	−198.43	−0.29 + j0.098
0	∞	−90	
∞	0	−270	


$G_R(j\omega)$
−1.8
−1.6
−1.4
−1.2
−1.0
−0.8
−0.6
−0.4
−0.2
−j0.2
−j0.4
−j0.6
−j0.8
−j1.0
−j1.2
−j1.4
$G_I(j\omega)$
$\phi_{gc} = -168°$
ω=1.0
ω=0.7
ω=0.6
ω=0.5
ω=0.45
ω=0.4
ω=0.35
Unity circle
Scale : 1 cm = 0.1 units
Gain margin, $K_g = 1/|G(j\omega_{pc})|$
$= 1/0.7 = 1.4286$
Phase margin, $\gamma = 180° + \phi_{gc}$
$= 180° - 168°$
$= 12°$

2) A system is given by

$$G(s) = \frac{1}{s^2(s+1)(s+10)}$$

Determine the magnitude and phase angle at zero and ∞ frequencies. Sketch the polar plot.

SOL:

Given $G(s) = \dfrac{1}{s^2(1+s)(10+s)}$

$$= \frac{0.1}{s^2(1+s)(1+0.1s)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{0.1}{(j\omega)^2(1+j\omega)(1+j0.1\omega)}$$

$$M\angle\phi = |G(j\omega)| \angle G(j\omega)$$

$$= \frac{0.1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(0.1\omega)^2}} \angle -180 - \tan^{-1}\omega - \tan^{-1}(0.1\omega)$$

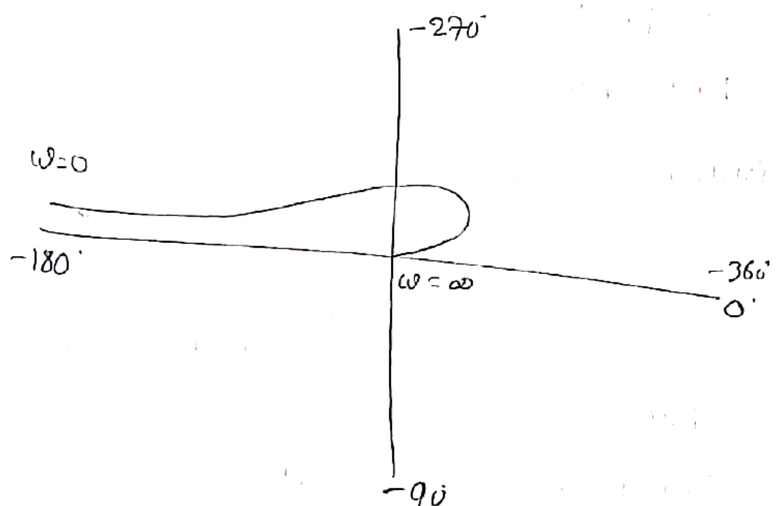
At $\omega = 0$, $M\angle\phi = \infty\angle -180^\circ$

At $\omega = \infty$, $M\angle\phi = 0\angle -360^\circ$ $(\omega_{c_1} = 1\ \text{rad/sec}\ \&\ \omega_{c_2} = 10\ \text{rad/sec})$

For various values of ω, the magnitude M and the phase angle ϕ are calculated and are given below

ω	M	ϕ	$M\cos\phi + jM\sin\phi$
0.5	0.36	-209.4	$-0.31 + j0.176$
1	0.07	-230.7	$-0.044 + j0.05$
1.5	0.024	-244.8	$-0.01 + j0.02$
2	0.01	-254.7	$-0.002 + j0.009$
2.5	0.006	-262.2	|
3	0.0033	-268.3	|
3.5	0.002	-273.3	|
4	0.0014	-277.7	|
4.5	0.0009	-281.7	so on

The polar plot is shown below.



$$G(j\omega) = \frac{0.1}{(j\omega)^2(1+j\omega)(1+j0.1\omega)}$$

$$= \frac{0.1}{-\omega^2(1+j1.1\omega - 0.1\omega^2)}$$

$$G(j\omega) = \frac{0.1}{(-\omega^2+0.1\omega^4) - j1.1\omega^3}$$

Rationalize the function

$$G(j\omega) = \frac{0.1}{(-\omega^2+0.1\omega^4) - j\,1.1\omega^3} \times \frac{(-\omega^2+0.1\omega^4) + j\,1.1\omega^3}{(-\omega^2+0.1\omega^4) + j\,1.1\omega^3}$$

$$= \frac{(-0.1\omega^2+0.01\omega^4) + j\,0.11\omega^3}{(-\omega^2+0.1\omega^4)^2 - (j\,1.1\omega^3)^2}$$

Seperating real & imaginary terms & equating real term to zero

$$\therefore -0.1\cancel{\omega^2} + 0.01\omega^{\cancel{4}\,2} = 0$$

$$0.01\omega^2 = 0.1$$

$$\omega = 3.1 \text{ rad/sec}$$

$\therefore$ The Polar plot can be intersect the imaginary axis at 3.1 rad/sec

3) Determine the gain crossover frequency, phase crossover frequency, gain margin and phase margin of a system whose open loop transfer function is

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

SOL:

To Find Phase cross over Frequency :- (ω_{PC})

Given $G(S) = \dfrac{1}{S(1+S)(1+2S)}$

put $S = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)(1+j\omega)(1+j2\omega)}$$

$$= \frac{1}{(j\omega)(1+j2\omega+j\omega-2\omega^2)} = \frac{1}{-3\omega^2 + j\omega(1-2\omega^2)}$$

At phase crossover frequency, $G(j\omega)$ is real i.e. imaginary part of $G(j\omega)$ is zero.

At $\omega = \omega_{PC}$, $\omega_{PC}(1-2\omega_{PC}^2) = 0$

$$\Rightarrow 1 = 2\omega_{PC}^2$$

$\therefore$ Phase crossover frequency, $\omega_{PC} = 0.707$ rad/sec.

To Find Gain Margin (K_g) :-

$$|G(j\omega)|_{\omega=\omega_{pc}} = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

$$= \frac{1}{0.707\sqrt{1+0.707^2}\sqrt{1+4(0.707)^2}}$$

$$= 0.67$$

$\therefore$ gain Margin, $K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{0.67}$

$$= 1.5$$

Gain Margin in db $= 20\log K_g = 20\log 1.5 = 3.5$ db.

To Find gain cross over Frequency (ω_{gc}) :-

$$G(j\omega) = \frac{1}{(j\omega)(1+j\omega)(1+j2\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

and $\phi = -90 - \tan^{-1}\omega - \tan^{-1}(2\omega)$

At Gain gain cross over frequency, $|G(j\omega)| = 1$

$$\therefore \frac{1}{\omega_{gc}\sqrt{1+\omega_{gc}^2}\sqrt{1+4\omega_{gc}^2}} = 1$$

Squaring on both sides

$$\omega_{gc}^2(1+\omega_{gc}^2)(1+4\omega_{gc}^2) = 1$$

$$\omega_{gc}^2(1+4\omega_{gc}^2+\omega_{gc}^2+4\omega_{gc}^4) = 1$$

$$\Rightarrow 4\omega_{gc}^6 + 5\omega_{gc}^4 + \omega_{gc}^2 = 1$$

By Trail and error method, $\omega_{gc} = 0.57$ rad/sec.

{∵ sub. different values of ω, at what value, $|G(j\omega)| = 1$ approx.}

To Find Phase Margin (γ) :-

at $\omega = \omega_{gc} = 0.57$ rad/sec, $\phi_{gc} = -90 - \tan^{-1}\omega_{gc} - \tan^{-1}(2\omega_{gc})$

$$= -90 - \tan^{-1}(0.57) - \tan^{-1}(2\times0.57)$$

$$\phi_{gc} = -168^\circ$$

∴ Phase Margin, $\gamma = 180 - 168^\circ$

$$\gamma = 12^\circ$$

4) The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{1}{s(1+s)^2}$$

Sketch the polar plot and determine GM & PM.

SOL:

Given $G(s) = \dfrac{1}{s(1+s)^2} = \dfrac{1}{s(1+s)(1+s)}$

Put $s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)(1+j\omega)(1+j\omega)}$$

$$\therefore M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+\omega^2}} = \frac{1}{\omega(1+\omega^2)}$$

$$= \frac{1}{\omega+\omega^3}$$

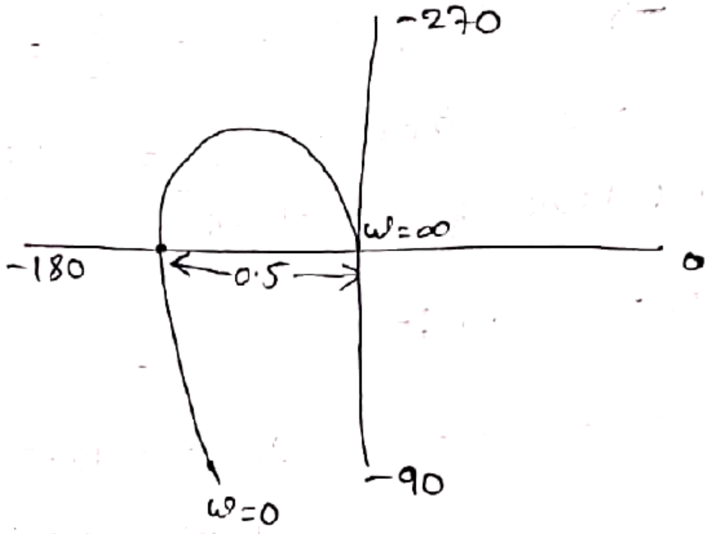
$$\phi = \angle G(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}\omega$$

$$= -90 - 2\tan^{-1}\omega$$

at $\omega = 0 \Rightarrow M\angle\phi = \infty\angle -90$

$\omega = \infty \Rightarrow M\angle\phi = 0\angle -270$

∴ The Polar plot is shown in the fig.



To Find ω_{pc} and GM :-

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{-2\omega^2 + j\omega(1-\omega^2)}$$

at $\omega = \omega_{pc}$, $G(j\omega)$ is real i.e. imaginary part is zero.

$$\therefore \omega_{pc}(1-\omega_{pc}^2) = 0$$

$$1 = \omega_{pc}^2$$

$\therefore$ Phase crossover Frequency, $\omega_{pc} = 1$ rad/sec

at $\omega = \omega_{pc} = 1$ rad/sec, $|G(j\omega)| = \frac{1}{\omega(1+\omega^2)}$

$$= \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$\therefore$ Gain Margin, $K_g = \frac{1}{|G(j\omega)|} = \frac{1}{0.5} = 2$

Gain Margin in db $= 20 \log K_g = 20 \log 2$

$$= 6 \text{ db}$$

To Find ω_{gc} and PM :-

At $\omega = \omega_{gc}$, $|G(j\omega)| = 1$

$$\frac{1}{\omega_{gc}(1+\omega_{gc}^2)} = 1$$

$$\Rightarrow \omega_{gc} + \omega_{gc}^3 = 1$$

By solving, $\omega_{gc} = 0.68$ rad/sec.

$\therefore$ Gain crossover Frequency, $\omega_{gc} = 0.68$ rad/sec.

$\therefore$ at $\omega = \omega_{gc} = 0.68$ rad/sec, $\phi_{gc} = -90 - 2\,\mathrm{Tan}^{-1}\omega_{gc}$

$$\phi_{gc} = -90 - 2\,\mathrm{Tan}^{-1}(0.68)$$
$$= -158.4^\circ$$

$\therefore$ Phase Margin, $\gamma = 180 + \phi_{gc}$

$$= 180 - 158.4 =$$
$$\underline{\gamma = 21.6^\circ}$$

5) The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s(1+0.5s)(1+4s)}$$

Sketch the polar plot and determine the value of K so that (i) Gain margin is 20 db and (ii) Phase margin is 30^0.

SOL:

Given that, G(s) = K/s (1+0.5s) (1+4s)

The polar plot is sketched by taking K = 1.

Put K = 1 and s = jω in G(s).

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

The corner frequencies are ω_{c1} = 1/4 = 0.25 rad/sec and ω_{c2} = 1/0.5 = 2 rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+(0.5\omega)^2}\ \angle\tan^{-1}0.5\omega\ \sqrt{1+(4\omega)^2}\ \angle\tan^{-1}4\omega}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}\ \angle(-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega$$

TABLE-1 : Magnitude and Phase of G(jω) at Various Frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
|G(jω)|	2.11	1.3	0.87	0.61	0.35	0.22	0.15
∠G(jω) deg	−149	−159	−167	−174	−184	−193	−199

TABLE-2 : Real part and Imaginary parts of G(jω) at Various Frequencies

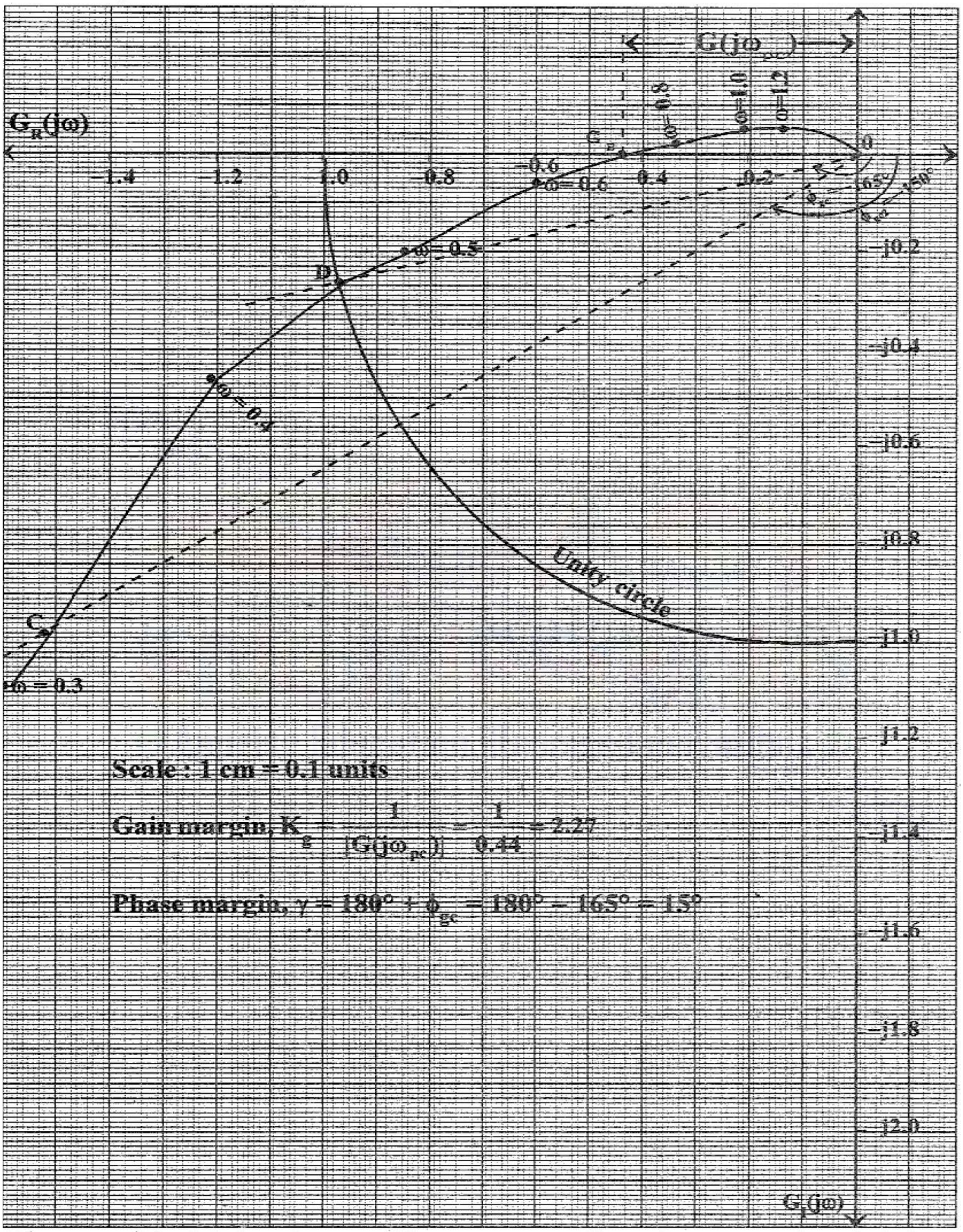
ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$G_R(j\omega)$	–1.8	–1.21	–0.85	–0.61	–0.35	–0.21	–0.14
$G_I(j\omega)$	–1.09	–0.47	–0.2	–0.06	0.02	0.05	0.05

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.44 = 2.27$

Gain margin in db = 20 log 2.27 = 7.12 db

Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 165° = 15°$



Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. From the polar plot, $G_B = 0.44$. The gain margin of 7.12 db with K = 1 has to be increased to 20 db and so K has to be decreased to a value less than one.

Let G_A be the gain at –180° for a gain margin of 20 db.

$$\text{Now, } 20 \log \frac{1}{G_A} = 20$$

$$\log \frac{1}{G_A} = \frac{20}{20} = 1$$

$$\frac{1}{G_A} = 10^1 = 10$$

$$\therefore G_A = \frac{1}{10} = 0.1$$

The value of K is given by, $K = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

Case (ii)

With K = 1, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°.

$$\therefore 30° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 30° - 180° = -150°$$

In the polar plot the –150° line cuts the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 2.04$ and $G_D = 1$

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{2.04} = 0.49$$

NYQUIST PLOT

Nyquist stability criterion relates the location of of the roots of the characteristic eq. to the open loop frequency response of the system.

The Nyquist stability criterion is based on a theorem of complex variables due to cauchy known as Principle of argument.

Introduction :-

Let F(S) be a function which is expressed two polynomials in s and is given by

$$F(S) = \frac{(S-Z_1)(S-Z_2)(S+Z_3)\ldots\ldots(S-Z_m)}{(S-P_1)(S-P_2)(S-P_3)\ldots\ldots(S-P_n)} \quad \text{——— (1)}$$

The function has m zeros and n poles.

Let s be a complex variable represented by $s = \sigma + j\omega$ on the complex s-plane. Then the function F(S) is also complex and let $F(S) = u + jv$ and represented on the complex F(S) plane.

From eq.(1) the function F(S) is analytic for every point s in the s-plane, there exists a corresponding point F(S) in F(S) plane. Hence it can be concluded that the function F(S) maps the points in the s-plane into F(S)-plane.

Note :- A function is analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function or it derivatives does not exist are called singular points.

Mapping :-

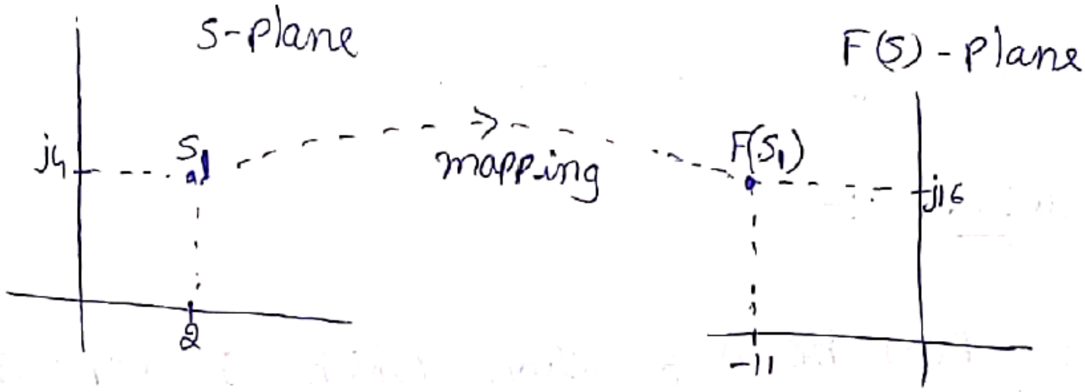
Let $F(S) = S_1^2 + 1$

if $S_1 = 2 + j4$

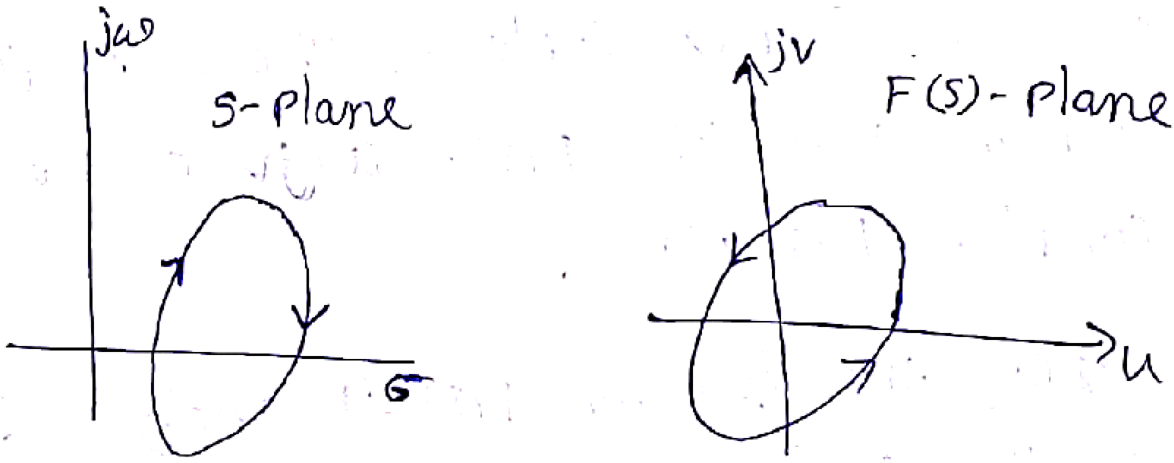
then $F(S) = F(2+j4) = F(S_1)$

$= (2+j4)^2 + 1 = -11 + j16$

The mapping can be shown below.

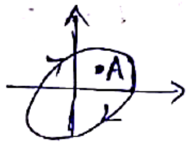


Since any no. of points of analycity in the S-plane can be mapped into F(S) plane, it can be concluded that for a contour in the S-plane which does not go through any point, there exist a corresponding contour in the F(S) plane as shown in the following fig.



Encircled :- A Point is said to be encircled by a closed path if it is found inside the path.

Ex :-

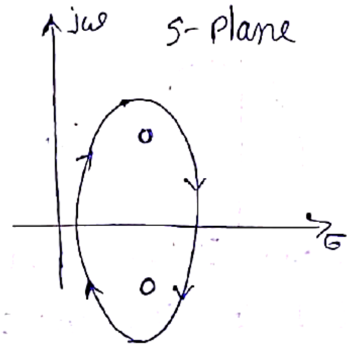


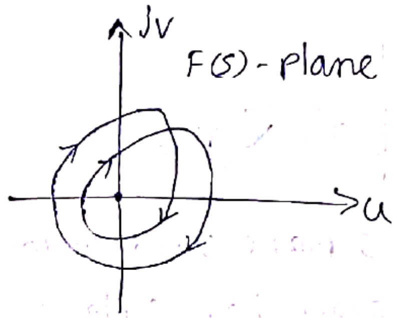
The Point A encircled in the clockwise direction.

Then there exist a relationship between the enclosure of poles and zeros by the s-plane closed contour and no. of encirclements of the origin of F(S) plane by the corresponding F(S)- plane contour.

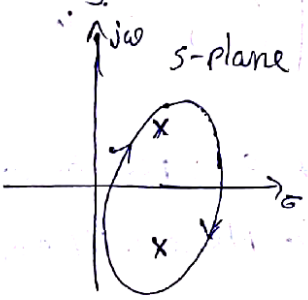
Important Points :-

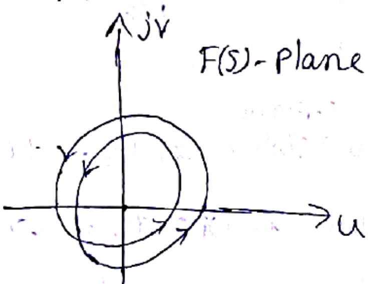
1) If s-plane closed contour encloses (enclosure) Z no. of zeros in the right half s-plane then the corresponding contour in F(S)- plane will encircle the origin of F(S)- plane Z times in the clockwise direction.



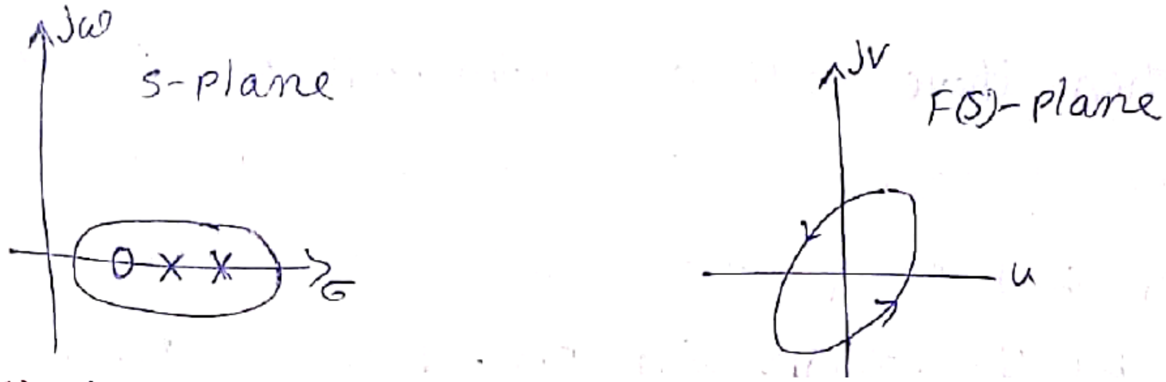


2) If s-plane closed contour encloses P no. of poles in the right half s-plane then the corresponding contour in F(S) - plane will encircle the origin of F(S)- plane P times in anti clockwise direction.

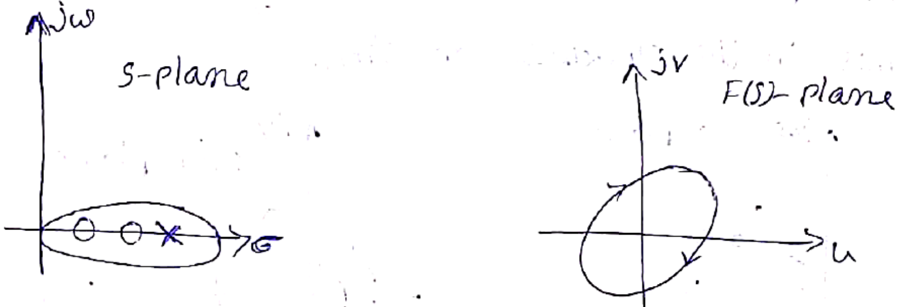




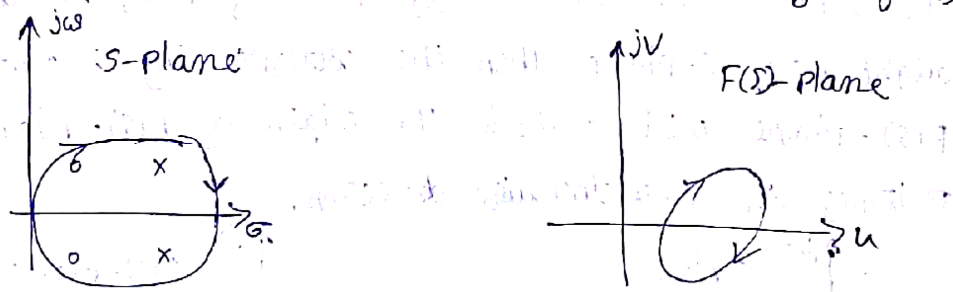
3) If s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $P > Z$, then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane $(P-Z)$ times in anticlockwise direction.



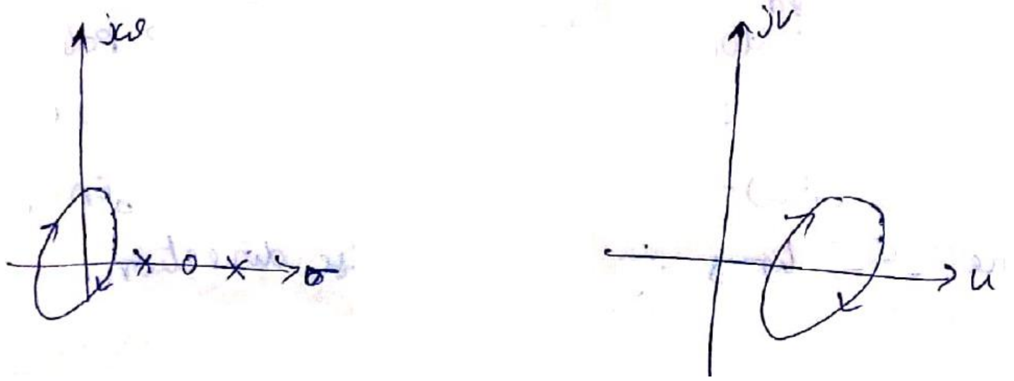
4) If s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $Z > P$, then the corresponding contour in F(s) plane will encircle the origin of F(s)-plane $(Z-P)$ times in clockwise direction.



5) If the s-plane closed contour encloses Z zeros and P poles in the right half s-plane and if $P = Z$, then the corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.



6) If the s-plane closed contour does not enclose any pole or zero then the corresponding contour in F(S)-plane will not encircle the origin of F(S)-plane.



PRINCIPLE OF ARGUMENT

The relation between the enclosure of poles and zeros of F(S) lying on right half s-plane by s-plane contour and the encirclements of the origin of F(S)-plane by the corresponding F(S)-plane contour is called principle of argument.

Let F(S) is a single valued rational function and is analytic in a given region in the s-plane except at some points. Now, if an arbitrary closed contour is chosen in s-plane, so that F(S) is analytic at every point on the closed contour in the s-plane then the corresponding F(S)-plane contour mapped in the F(S)-plane will encircle the origin N times in anticlockwise direction where N is the difference between the no. of poles and no. of zeros of F(S) that are encircled by the chosen closed contour in s-plane.

i.e. $$N = P - Z$$

where N = No. of encirclements of the origin made by the contour of F(S)- Plane.

Z = no. of Zeros of F(S) lying on right half s-plane and encircled by the s-plane closed contour.

P = No. of Poles of F(S) lying on right half s-plane and encircled by the s-plane closed contour.

If N is +ve, the encirclement of the origin of F(S)- plane will be in anticlockwise direction.

If N is −ve, the encirclement of the origin of F(S) - plane will be in clockwise direction.

If N is zero, then the Poles and zeros are equal and there will be no encirclement of the origin of F(S)-plane.

NYQUIST STABILITY CRITERIA

Consider the characteristic eq. of the system is

$$q(S) = 1 + G(S)H(S) = 0$$

Let $$G(S)H(S) = K\frac{(S+Z_1)(S+Z_2)\ldots\ldots(S+Z_m)}{(S+P_1)(S+P_2)\ldots\ldots(S+P_n)} \quad ; m \leq n$$

$$\therefore \quad q(S) = 1 + K\frac{(S+Z_1)(S+Z_2)\ldots\ldots(S+Z_m)}{(S+P_1)(S+P_2)\ldots\ldots(S+P_n)}$$

$$= \frac{(S+P_1)(S+P_2)\ldots\ldots(S+P_n) + K(S+Z_1)(S+Z_2)\ldots\ldots(S+Z_m)}{(S+P_1)(S+P_2)\ldots\ldots(S+P_n)}$$

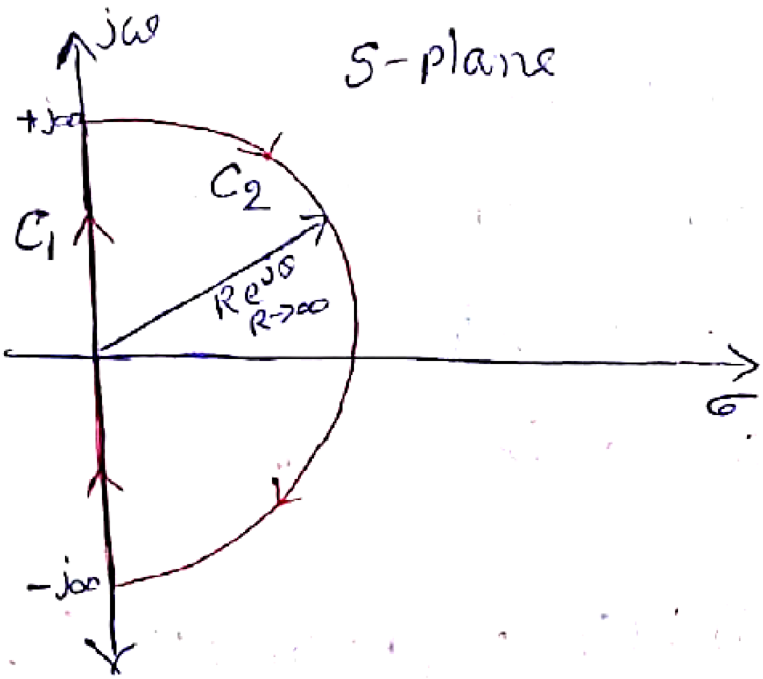
$$q(s) = \frac{(s+z'_1)(s+z'_2)\ldots\ldots(s+z'_m)}{(s+p_1)(s+p_2)\ldots\ldots(s+p_n)}$$

where $-z'_1, -z'_2, \ldots\ldots, -z'_n$ are the roots the characteristic eq and zeros of $q(s)$.

$-p_1, -p_2, \ldots\ldots, -p_n$ are the poles of $q(s)$ i.e. same as open loop poles of the system.

For the system to be stable, the roots of the characteristic eq. and hence zeros of $q(s)$ must lie in the left half of the s-plane.

In order to investigate the presence of any zero of $q(s) = 1+G(s)H(s)$ in the right half s-plane, consider a contour (C) which completely encloses the right half of the s-plane. Such a contour (C) is called Nyquist contour and is shown in the following fig.



The Nyquist contour is directed clockwise and comprises of an infinite line segment C_1 along the $j\omega$-axis and an arc C_2 of infinite radius.

Along C_1, Put $s = j\omega$ where ω varying from $-j\infty$ to $+j\infty$

Along C_2, Put $s = \lim_{R \to \infty} Re^{j\theta}$ where θ varying from $\frac{\pi}{2} \to 0 \to \left(-\frac{\pi}{2}\right)$.

The Nyquist contour encloses all the right half s-plane zeros and poles of $q(s) = 1 + G(s)H(s)$.

Let there are Z zeros and P poles of $q(s)$ in the right half s-plane. As s moves along the Nyquist contour in the s-plane, the closed contour Γ_q is traversed in the $q(s)$-plane which encloses the origin by $N = P - Z$ times in the counter clockwise direction.

For the system to be stable, there should be no zeros of $q(s) = 1 + G(s)H(s)$ in the right half s-plane.

i.e. $Z = 0$

then $N = P$

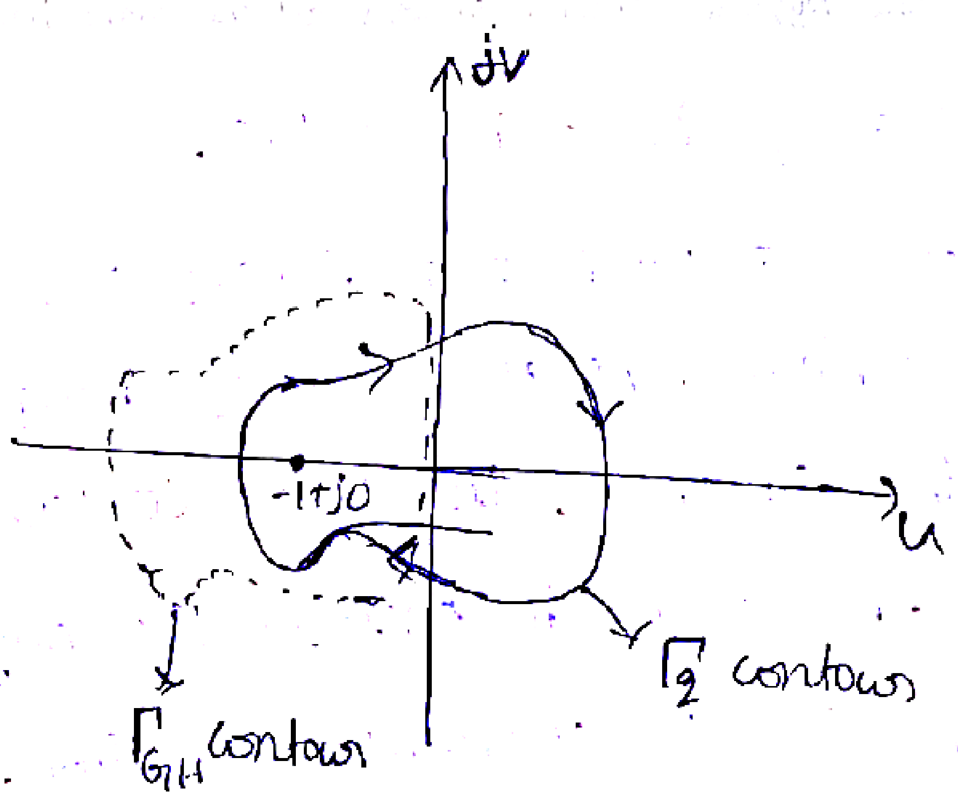
$\therefore$ For a closed loop system to be stable, the no. of counter clockwise encirclements of the origin of $q(s)$-plane by the contour Γ_q should equal to the no. of right half s-plane poles of $q(s)$, which are poles of open loop T.F $G(s)H(s)$.

If the open loop system is stable i.e. $P = 0$, the closed loop system is also stable if $N = P = 0$.
i.e. the net encirclements of the origin of $q(s)$-plane by Γ_q contour should be zero.

Then

$$G(s)H(s) = \left[1 + G(s)H(s)\right] - 1$$

∴ The contour Γ_{GH} of $G(s)\,H(s)$ corresponding to the Nyquist contour in the s-plane is same ~~of~~ as contour Γ_2 of $1+G(s)\,H(s)$ drawn from the same point $(-1+j0)$. Thus the encirclement of the origin by the contour Γ_2 is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH} and is shown in the fig.



STATEMENT OF NYQUIST STABILITY CRITERIA

If the contour Γ_{GH} of the open loop T.F $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1+j0)$ in the counterclockwise direction as many times as the no. of ~~ne~~ right half s-plane poles of $G(s)H(s)$, the closed loop system is stable.

The closed loop system is stable if the contour Γ_{GH} of $G(s)H(s)$ does not encircle $(-1+j0)$ point i.e. the net encirclement is zero.

For Nyquist stability criterion,

1. There is no encirclement of $(-1+j0)$ point. That means the system is stable if there are no poles of $G(s)H(s)$ in the right half s-plane. If there are poles on right half s-plane, then the system is unstable.

2. An anticlockwise encirclement of $(-1+j0)$ point. ~~That~~ In this case, the system is stable if the no. of anticlockwise encirclement is same as the no. of poles of $G(s)H(s)$ in the right half s-plane. If the no. of encirclements is not equal to the no. of poles on the right half- s-plane, then the system is unstable.

3. There is a clockwise encirclement of the $(-1+j0)$ point. In this case the system is always unstable.

The mapping of Nyquist contour into the contour Γ_{GH} is explained as follows.

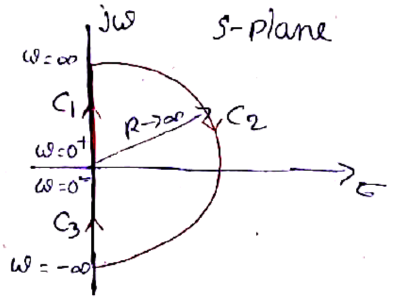
(i) The mapping of the imaginary axis is carried out by putting $s = j\omega$ in $G(s)H(s)$. This converts the mapping function into a frequency function of $G(j\omega)H(j\omega)$.

(ii) In physical systems $(m \le n)$, put $\lim_{R \to \infty} s = Re^{j\theta}$ in $G(s)H(s)$ = real constant. Thus the infinite arc of the Nyquist contour maps into a point on the real axis.

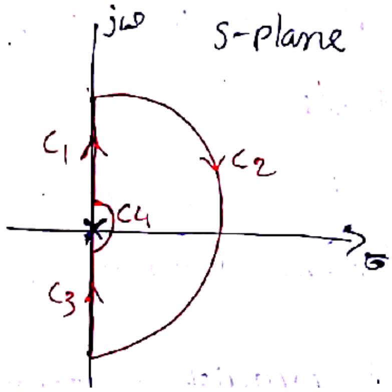
The complete contour Γ_{GH} is thus the polar plot of $G(j\omega)H(j\omega)$ with ω varying from $-\infty$ to ∞. This is usually called Nyquist plot or locus of $G(s)H(s)$.

Note :- The Nyquist contour for poles on imaginary axis is shown below.

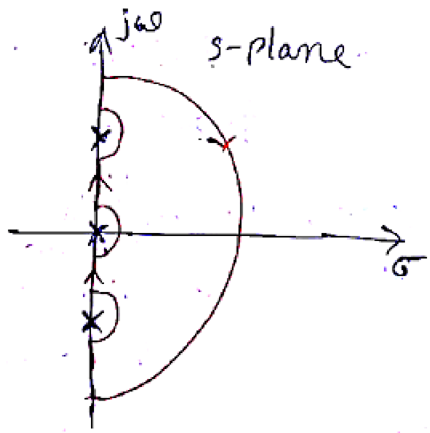
1) The Nyquist contour when there is no pole on imaginary axis is given by



2) The Nyquist contour when there are poles at origin is given by



3) The Nyquist contour when there are poles on imaginary axis and at origin is given by



PROBLEMS

1) Draw the Nyquist plot for the system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+2)(s+10)}$$

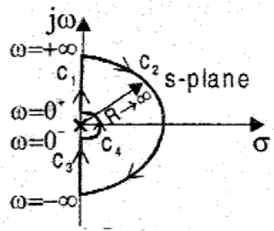
Determine the range of 'K' for which the closed loop system is stable.

SOL:

$$\text{Given that, } G(s)H(s) = \frac{K}{s\,(s+2)\,(s+10)} = \frac{K}{s \times 2\left(\frac{s}{2}+1\right) \times 10\left(\frac{s}{10}+1\right)} = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.



MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

Let $s = j\omega$.

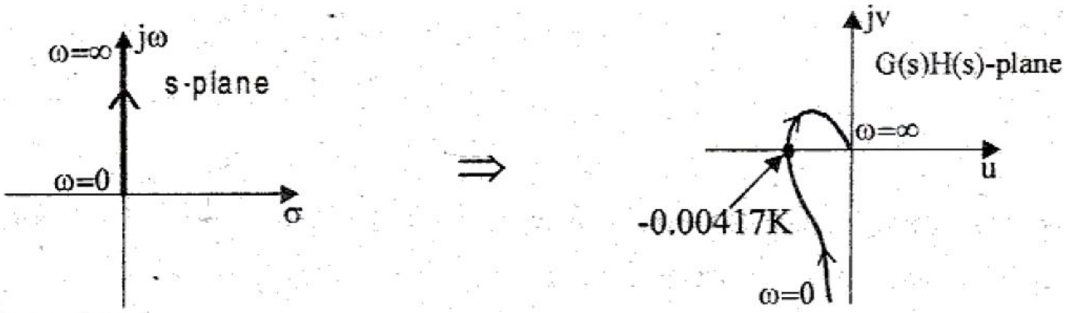
$$\therefore G(j\omega)H(j\omega) = \frac{0.05K}{j\omega\,(1+j0.5\omega)\,(1+j0.1\omega)} = \frac{0.05K}{j\omega\,(1+j0.6\omega-0.05\omega^2)} = \frac{0.05K}{-0.6\omega^2 + j\omega\,(1-0.05\omega^2)}$$

When the locus of $G(j\omega)H(j\omega)$ crosses real axis the imaginary term will be zero and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}(1-0.05\omega_{pc}^2) = 0 \quad \Rightarrow \quad 1-0.05\omega_{pc}^2 = 0 \quad \Rightarrow \quad \omega_{pc} = \sqrt{\frac{1}{0.05}} = 4.472 \text{ rad/sec}$$

$$\text{At } \omega = \omega_{pc} = 4.472 \text{ rad/sec}, \qquad G(j\omega)H(j\omega) = \frac{0.05K}{-0.6\omega^2} = -\frac{0.05K}{0.6 \times (4.472)^2} = -0.00417K$$

The open loop system is type-1 and third order system. Also it is a minimum phase system with all poles. Hence the polar plot of $G(j\omega)H(j\omega)$ starts at −90° axis at infinity, crosses real axis at −0.00417K and ends at origin in second quadrant. The section C_1 and its mapping are shown in fig



MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt} R e^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R e^{j\theta}$ and $R \to \infty$, the G(s)H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)\,H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 0.5s \times 0.1s} = \frac{K}{s^3}$$

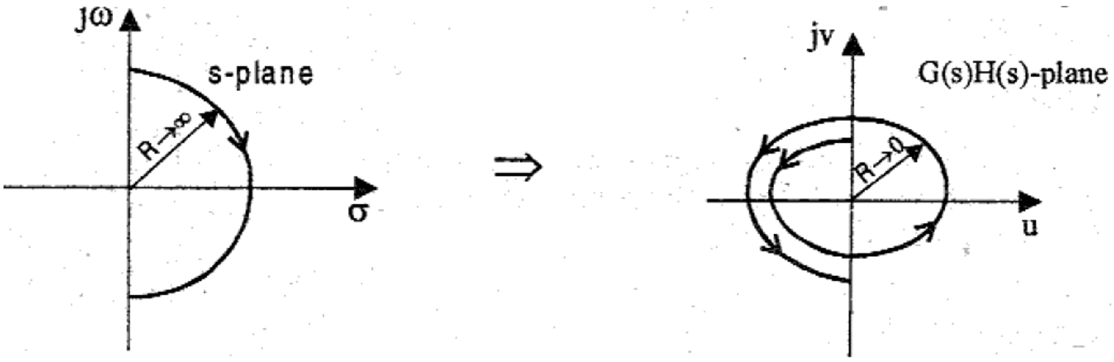
Let, $s = \underset{R\to\infty}{Lt} Re^{j\theta}$.

$$\therefore\ G(s)H(s)\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{s^3}\Big|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt} (Re^{j\theta})^3} = 0e^{-j3\theta}$$

When $\theta = \frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{-j3\frac{\pi}{2}}$ .....(1)

When $\theta = -\frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{+j3\frac{\pi}{2}}$ .....(2)

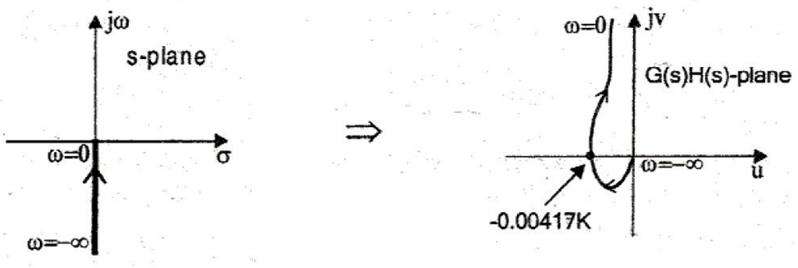
From the equations (1) and (2) we can say that section C_2 in s-plane is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-3\pi/2$ to $+3\pi/2$ as shown in fig



MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of G(jω)H(jω) as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of G(jω)H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig



MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s) H(s) can be approximated as shown below, [i.e., $(1+sT) \approx 1$].

$$G(s)H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 1 \times 1} = \frac{0.05K}{s}$$

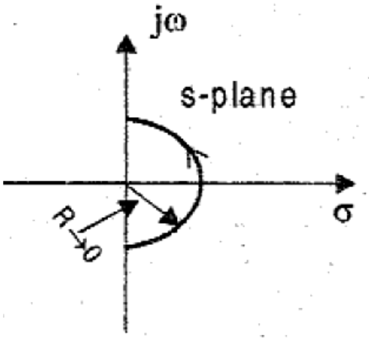
Let $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$.

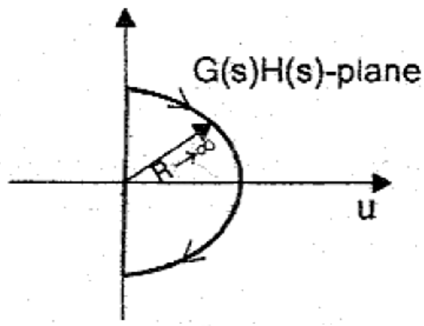
$$\therefore\ G(s)H(s)\Bigg|_{s=\underset{R\to 0}{Lt}Re^{j\theta}} = \frac{0.05K}{s}\Bigg|_{s=\underset{R\to 0}{Lt}Re^{j\theta}} = \frac{0.05K}{\underset{R\to 0}{Lt}(Re^{j\theta})} = \infty e^{-j\theta}$$

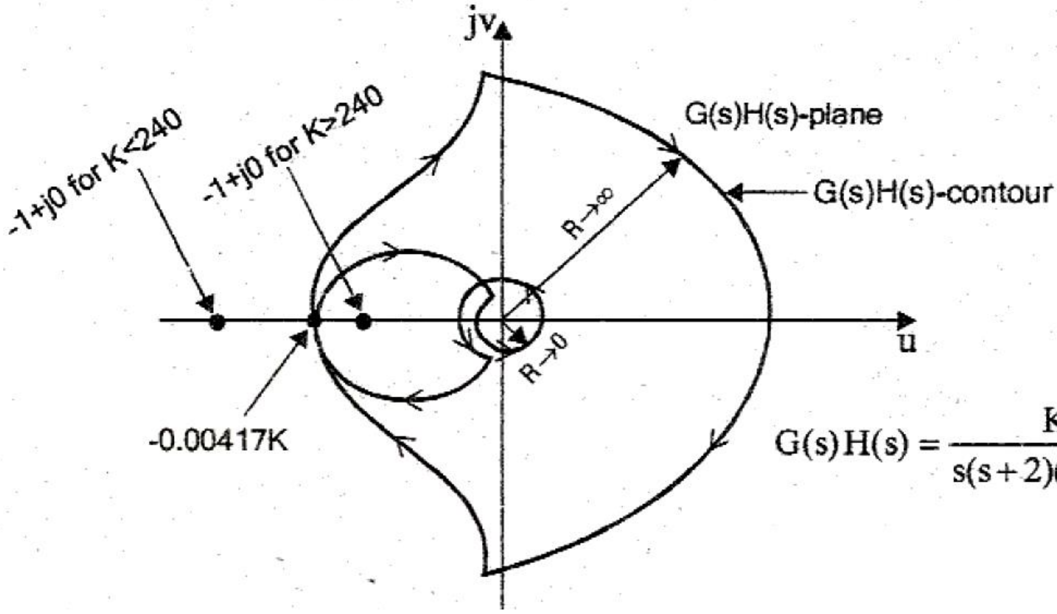
$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)H(s) = \infty e^{+j\frac{\pi}{2}} \qquad .....(3)$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)H(s) = \infty e^{-j\frac{\pi}{2}} \qquad .....(4)$$

From the equations (3) and (4) we can say that section C_4 in s-plane is mapped as a circular arc of infinite radius with argument (phase) varying from $+\pi/2$ to $-\pi/2$ as shown in fig







$$G(s)H(s) = \frac{K}{s(s+2)(s+10)}$$

STABILITY ANALYSIS

When, $-0.00417K = -1$, the contour passes through $(-1+j0)$ point and corresponding value of K is the limiting value of K for stability.

$$\therefore \text{Limiting value of } K = \frac{1}{0.00417} = 240$$

When K < 240

When K is less than 240, the contour crosses real axis at a point between 0 and $-1+j0$. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is not encircled. Also the open loop transfer function has no poles on the right half of s-plane. Therefore the closed loop system is stable.

When K > 240

When K is greater than 240, the contour crosses real axis at a point between $-1+j0$ and $-\infty$. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is encircled in clockwise direction two times. [Since there are two clockwise encirclement and no right half open loop poles, the closed loop system has two poles on right half of s-plane]. Therefore the closed loop system is unstable.

RESULT

The value of K for stability is $0 < K < 240$

2) Draw the Nyquist plot and comment on stability for the system

$$G(s) = \frac{s + 0.25}{s^2(s+1)(s+0.5)}$$

SOL:

Put $s = j\omega$

$$G(j\omega)\,H(j\omega) = \frac{0.25 + j\omega}{(j\omega)^2\,(1+j\omega)\,(0.5+j\omega)}$$

$$\therefore \quad M = |G(j\omega)\,H(j\omega)| = \frac{\sqrt{0.25^2+\omega^2}}{\omega^2\sqrt{1+\omega^2}\,\sqrt{0.5^2+\omega^2}}$$

and $\phi = \angle G(j\omega)\,H(j\omega) = \tan^{-1}\frac{\omega}{0.25} - 180 - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{0.5}$

At $\omega = 0$, $M\angle\phi = \infty\angle{-180^\circ}$

$\omega = \infty$, $M\angle\phi = 0\angle{-270}$

Intersection Point with real axis :-

$$G(j\omega)\ H(j\omega) = \frac{0.25+j\omega}{(j\omega)^2\,(1+j\omega)\,(0.5+j\omega)}$$

Rationalising the above term.

$$G(j\omega)\ H(j\omega) = \frac{0.25+j\omega}{-\omega^2(1+j\omega)(0.5+j\omega)} \cdot \frac{1-j\omega}{1-j\omega} \cdot \frac{0.5-j\omega}{0.5-j\omega}$$

$$= \frac{(0.25+j\omega)\,(0.5-1.5j\omega-\omega^2)}{-\omega^2(1+\omega^2)\,(0.25+\omega^2)}$$

$$= \frac{0.125+1.25\omega^2}{-\omega^2(1+\omega^2)(0.25+\omega^2)} + j\,\frac{\omega(0.125-\omega^2)}{-\omega^2(1+\omega^2)(0.25+\omega^2)}$$

Equating imaginary part to zero

$$\Rightarrow \quad \omega\,(0.125-\omega^2) = 0$$

$$\omega^2 = 0.125$$

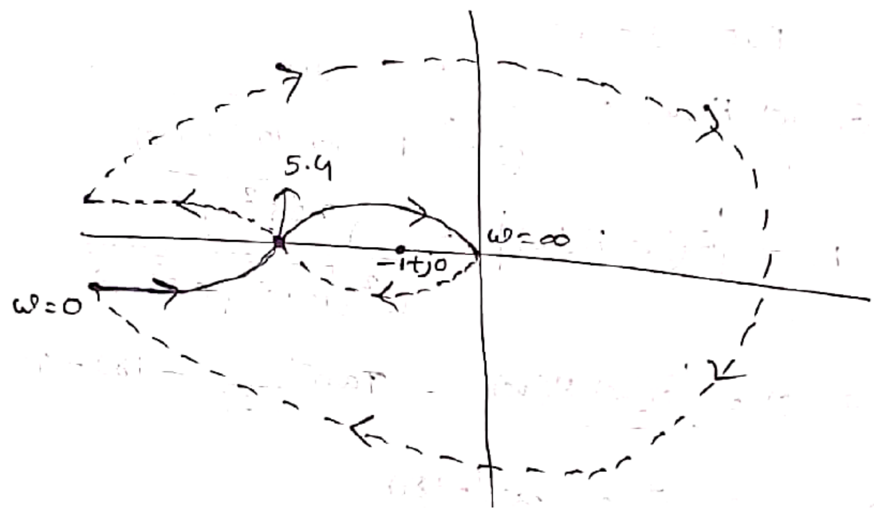
$$\omega = 0.35 \text{ rad/sec}$$

At $\omega = 0.35$, $$|G(j\omega)\ H(j\omega)| = \frac{\sqrt{0.25^2+0.35^2}}{(0.35)^2\sqrt{1+0.35^2}\,\sqrt{0.5^2+0.35^2}}$$

$$= 5.4$$

∴ The Nyquist Plot intersects the real axis at -5.4

The complete Nyquist Plot is shown in the fig.



∴ The no. of encirclements of the point (-1+j0) is

$N = -2$ (clockwise direction)

i.e. $P = 0$

∴ $Z = 2$

∴ the closed loop system is unstable.

[$GM = 20 \log \frac{1}{5\cdot 4} = -14.6$

GM is -ve ; hence system is unstable]

Note :- For stable, N is +ve

For unstable, N is -ve.

3) Draw the Nyquist plot and comment on stability for the system

$$G(s)H(s) = \frac{K}{(1+T_1 s)(1+T_2 s)}$$

SOL:

The open-loop sinusoidal transfer function is

$$G(j\omega)H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

Rationalizing,

$$G(j\omega)H(j\omega) = \frac{K(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1+j\omega T_1)(1-j\omega T_2)(1+j\omega T_2)}$$

$$= \frac{K[1-\omega^2 T_1 T_2] - jK\omega(T_1+T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} = \frac{K(1-\omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} - \frac{jK\omega(T_1+T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$$

Along the segment (C_1) of the Nyquist contour on the $j\omega$-axis, s varies from $-j\infty$ to $+j\infty$.

At $\omega = -\infty$,

$$G(j\omega)H(j\omega)=\frac{K[1-(-\infty)^2T_1T_2]}{[1+(-\infty)^2T_1^2][1+(-\infty)^2T_2^2]}-\frac{jK(-\infty)(T_1+T_2)}{[1+(-\infty)^2T_1^2][1+(-\infty)^2T_2^2]}=-0+j0$$

At $\omega = 0^-$,

$$G(j\omega)H(j\omega)=\frac{K[1-(-0)^2T_1T_2]}{[1+(-0)^2T_1^2][1+(-0)^2T_2^2]}-\frac{jK(-0)(T_1+T_2)}{[1+(-0)^2T_1^2][1+(-0)^2T_2^2]}=K+j0$$

At $\omega = 0^+$,

$$G(j\omega)H(j\omega)=\frac{K[1-(+0)^2T_1T_2]}{[1+(+0)^2T_1^2][1+(+0)^2T_2^2]}-\frac{jK(+0)(T_1+T_2)}{[1+(+0)^2T_1^2][1+(+0)^2T_2^2]}=K-j0$$

At $\omega = +\infty$,

$$G(j\omega)H(j\omega)=\frac{K[1-(+\infty)^2T_1T_2]}{[1+(+\infty)^2T_1^2][1+(+\infty)^2T_2^2]}-\frac{jK(+\infty)(T_1+T_2)}{[1+(+\infty)^2T_1^2][1+(+\infty)^2T_2^2]}=-0-j0$$

So, we get four points to draw an approximate Nyquist plot.

The infinite semi-circular arc of the Nyquist contour (segment C_2) of Figure mapped like this.

Along the semi-circular arc,

$$s = \underset{R\to\infty}{Re^{j\phi}}$$

where ϕ varies from $\pi/2$ through 0° to $-\pi/2$. Therefore,

$$G(s)H(s)=\underset{R\to\infty}{\text{Lt}}\ \frac{K}{(1+Re^{j\phi}T_1)(1+Re^{j\phi}T_2)}$$

$$=\underset{\substack{R\to\infty\\ \phi\to\frac{\pi}{2}\text{ to }-\frac{\pi}{2}}}{\text{Lt}}\ \frac{K}{R^2e^{j2\phi}T_1T_2}=0.e^{-j2\phi}=0\angle-2\phi$$

So the magnitude is zero and the phase varies from $-2\times(\pi/2)$ to $-2\times(-\pi/2)$, i.e. from −180° to +180°. So the infinite semi-circular arc is mapped onto a point at the origin joining the $\omega = +\infty$ and $\omega = -\infty$ points in the $q(s)$-plane.

The point of intersection of the Nyquist plot with the imaginary axis is obtained by equating the real part of $G(j\omega)H(j\omega)$ to zero. Therefore,

$$\frac{K(1-\omega^2T_1T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}=0$$

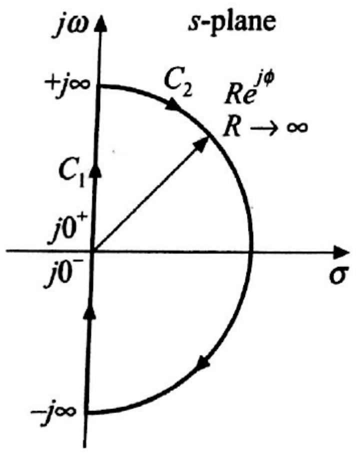
$\therefore$
$$1-\omega^2T_1T_2=0$$

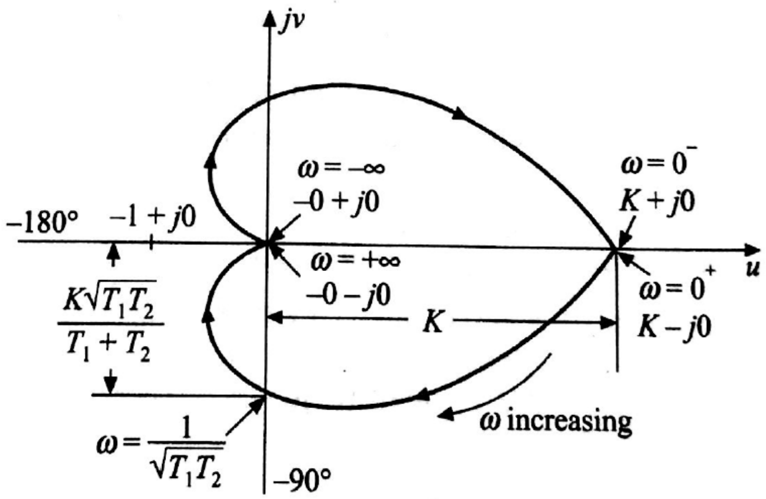
or
$$\omega=\frac{1}{\sqrt{T_1T_2}}$$

The value of $G(j\omega)H(j\omega)$ at that point of intersection is obtained by substituting this value of $\omega = \frac{1}{\sqrt{T_1T_2}}$ in the imaginary part, i.e.

$$G(j\omega)H(j\omega)\left(\text{at } \omega = \frac{1}{\sqrt{T_1T_2}}\right) = -j\frac{K\cdot\frac{1}{\sqrt{T_1T_2}}(T_1+T_2)}{\left[1+\left(\frac{1}{\sqrt{T_1T_2}}\right)^2 T_1^2\right]\left[1+\left(\frac{1}{\sqrt{T_1T_2}}\right)^2 T_2^2\right]} = -j\frac{K\sqrt{T_1T_2}}{T_1+T_2}$$

Based on the above information, an approximate Nyquist plot is drawn as shown in Figure From Figure it can be observed that the Nyquist plot of $G(j\omega)H(j\omega)$ does not encircle the $(-1+j0)$ point of $q(s)$ plane for any positive values of K, T_1 and T_2. Therefore, the system is stable for all positive values of K, T_1 and T_2.





4) Draw the Nyquist plot and comment on stability for the system

$$G(s)H(s) = \frac{(6s+1)}{s^2(s+1)(3s+1)}$$

SOL:

The given open-loop system has a double pole at the origin. The Nyquist contour is, therefore, indented to bypass the origin as shown in Figure (a). The mapping of the Nyquist contour is obtained as follows.

The given open-loop transfer function in sinusoidal form is

$$G(j\omega)H(j\omega) = \frac{(j6\omega+1)}{(j\omega)^2(j\omega+1)(j3\omega+1)} = \frac{(j6\omega+1)(1-j\omega)(1-j3\omega)}{-\omega^2(1+\omega^2)(1+9\omega^2)}$$

$$= \frac{[1+21\omega^2]}{-\omega^2(1+\omega^2)(1+9\omega^2)} - j\frac{(2-18\omega^2)}{\omega(1+\omega^2)(1+9\omega^2)}$$

Along the segment (C_1) of the Nyquist contour on the $j\omega$-axis, s varies from $-j\infty$ to $+j\infty$.

At $\omega = -\infty$, $$G(j\omega)H(j\omega) = -0 - j0$$

At $\omega = 0^-$, $$G(j\omega)H(j\omega) = -\infty + j\infty$$

At $\omega = 0^+$, $$G(j\omega)H(j\omega) = -\infty - j\infty$$

At $\omega = +\infty$, $$G(j\omega)H(j\omega) = -0 + j0$$

So, we get four points to draw an approximate Nyquist plot. The infinite semicircular arc of the Nyquist contour (segment C_2) of Figure (a) represented by $s = \underset{\epsilon \to 0}{\text{Lt}} \epsilon e^{j\theta}$ (where θ varies from $-90°$ through $0°$ to $+90°$) is mapped into

$$\underset{\epsilon \to 0}{\text{Lt}} \left[\frac{6\epsilon e^{j\theta} + 1}{\epsilon^2 e^{j2\theta}(\epsilon e^{j\theta} + 1)(3\epsilon e^{j\theta} + 1)} \right] = \underset{\epsilon \to 0}{\text{Lt}} \left(\frac{1}{\epsilon^2 e^{j2\theta}} \right) = \infty e^{-j2\theta}$$

$$= \infty\ (\angle 180° \to \angle 0° \to \angle -180°)$$

that is, into a semicircle of infinite radius extending from $+180°$ through $0°$ to $-180°$ as shown in Figure (b).

The infinite semicircle of the Nyquist contour of Figure (a) represented by $s = \underset{R \to \infty}{\text{Lt}} Re^{j\phi}$ (ϕ varies from $+90°$ through $0°$ to $-90°$) is mapped into

$$\underset{R \to \infty}{\text{Lt}} \left[\frac{6Re^{j\phi} + 1}{(R^2 e^{j2\phi})(Re^{j\phi} + 1)(3Re^{j\phi} + 1)} \right] = \underset{R \to \infty}{\text{Lt}} \frac{6Re^{j\phi}}{3R^4 e^{j4\phi}} = \underset{R \to \infty}{\text{Lt}} \frac{2}{R^3 e^{j3\phi}} = 0e^{-j3\phi}$$

$$= 0\ (\angle -270° \to \angle 0° \to \angle +270°)$$

The point of intersection of the Nyquist plot on the real axis is obtained by equating the imaginary part to zero, i.e.

$$\frac{(2 - 18\omega^2)}{-\omega(1 + \omega^2)(1 + 9\omega^2)} = 0$$

i.e. $$2 - 18\omega^2 = 0$$

or $$\omega^2 = 1/9$$

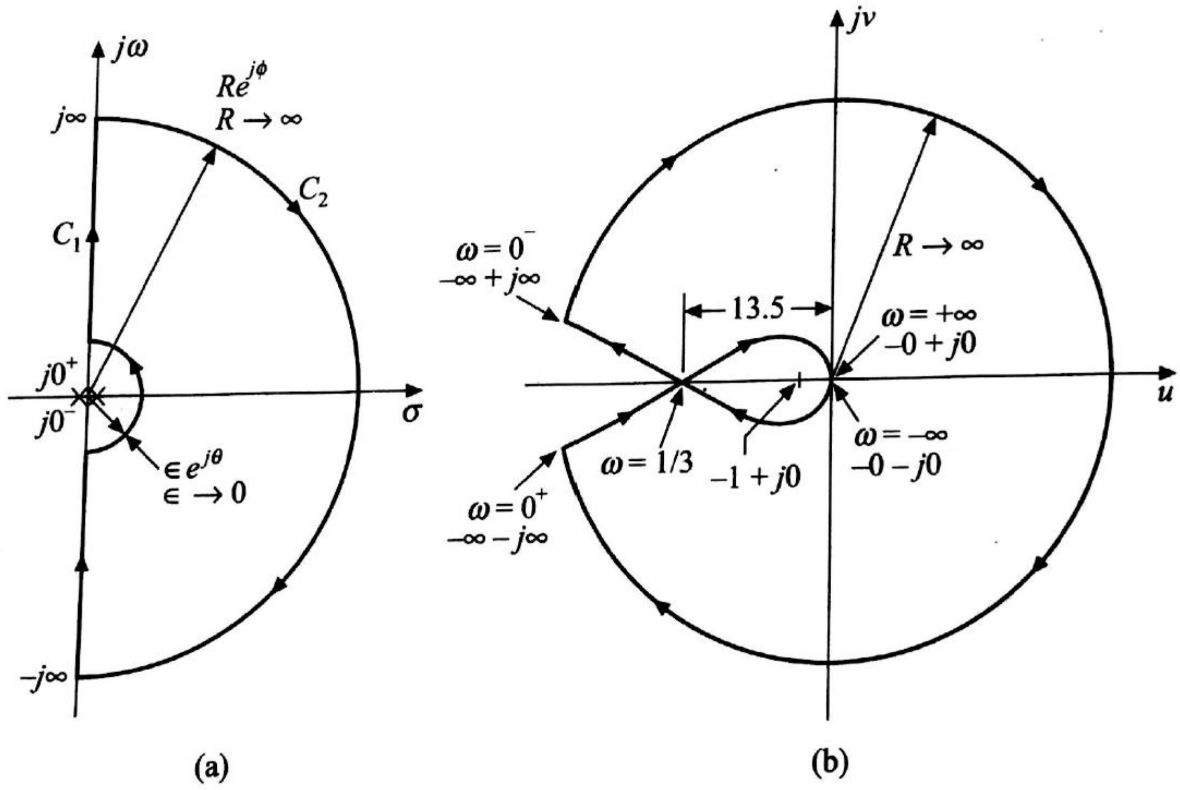
or $$\omega = 1/3 \text{ rad/s}$$

The value of $G(j\omega)H(j\omega)$ at $\omega = 1/3$ is obtained by substituting this value of ω in the real part of $G(j\omega)H(j\omega)$, i.e.

$$\frac{1 + 21\omega^2}{-\omega^2(1 + \omega^2)(1 + 9\omega^2)} = \frac{1 + 21/9}{(-1/9)(1 + 1/9)(1 + 9/9)} = \frac{30/9}{10 \times 2/9 \times 9} = -13.5$$

So, the Nyquist plot crosses the real axis at -13.5.

Based on the above information, an approximate Nyquist plot drawn for the Nyquist path shown in Figure (a) is shown in Figure (b). From this plot we can observe that, the Nyquist plot of $G(s)H(s)$ encircles the $(-1 + j0)$ point twice in the clockwise direction. Therefore, $N = -2$. The given open-loop transfer function $G(s)H(s)$ has no poles in the right-half of the s-plane. So, $P = 0$. Thus, $-2 = 0 - Z$ or $Z = 2$. Hence two zeros of $q(s)$ lie in the right-half of the s-plane. So the closed-loop system is unstable.



5) Draw the Nyquist plot and comment on stability for the system

$$G(s)\ H(s) = \frac{K(s+10)(s+2)}{(s+0.5)(s-2)}$$

SOL:

The Nyquist path consists of the entire jω axis and the infinite semicircle enclosing the right half of s-plane.

For $s = j\omega$ and $\omega \rightarrow 0$ to ∞

$$G(s)\ H(s) = \frac{K(j\omega+10)(j\omega+2)}{(j\omega+0.5)(j\omega-2)}$$

for $\omega = 0$

$$G(s)\ H(s) = \frac{20k}{-1} = -20k = 20K\angle 180$$

for $\omega = \infty$

$$G(s)\ H(s) = \underset{\omega \to 0}{\text{lt}} \frac{K(j\omega)^2\left(1+\frac{10}{j\omega}\right)\left(1+\frac{2}{j\omega}\right)}{(j\omega)^2\left(1+\frac{0.5}{j\omega}\right)\left(1-\frac{2}{j\omega}\right)}$$

$$= K$$

To find the possible crossing of negative real axis,

$$\operatorname{Im} G(j\omega) H(j\omega) = 0$$

$$\operatorname{Im} \frac{K(j\omega+10)(j\omega+2)(-j\omega+0.5)(-j\omega-2)}{(\omega^2+0.25)(\omega^2+4)} = 0$$

$$\operatorname{Im} (-\omega^2 + 20 + 12j\omega)(-\omega^2 - 1 + 1.5j\omega) = 0$$

$$-1.5\omega^2 + 30 - 12 - 12\omega^2 = 0$$

$$\omega^2 = \frac{18}{13.5} = \frac{4}{3}$$

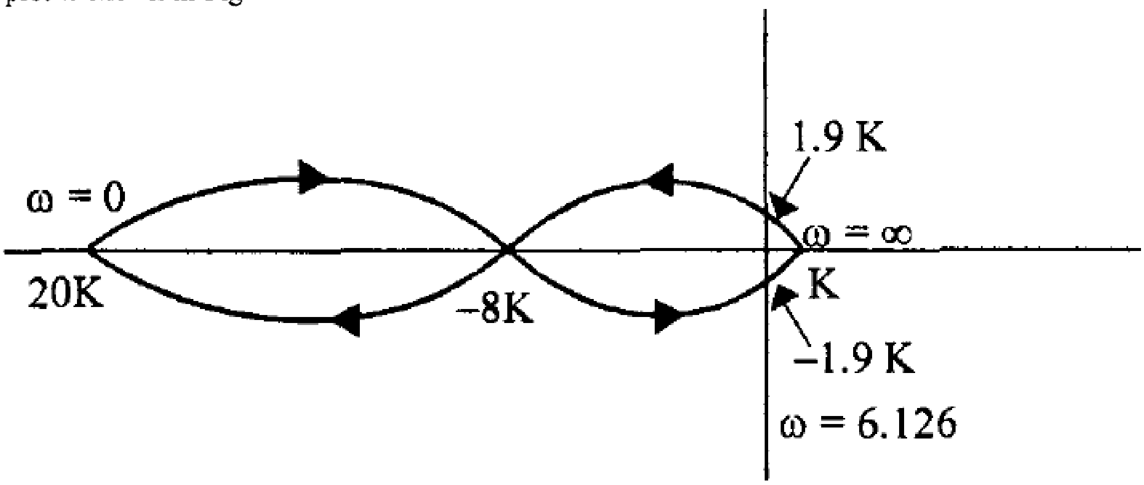
$$\omega = 1.1547 \text{ rad/sec}$$

$$\operatorname{Re}[G(j\omega) H(j\omega)]_{\omega = 1.1547} = K \left. \frac{-(20-\omega^2)(1+\omega^2) - 18\omega^2}{(\omega^2 + .25)(\omega^2+4)} \right|_{\omega = 1.1547}$$

$$= -8K$$

Hence the Nyquist plot crosses the negative real axis at $-8K$ for $\omega = 1.1547$ rad/sec.

The infinite semicircle of Nyquist path maps into the origin of GH plane. The negative imaginary axis maps into a mirror image of the Nyquist plot of the positive $j\omega$ axis. Hence the complete Nyquist plot is shown in Fig.



By equating the real part of $G(j\omega) H(j\omega)$ to zero, we can get the crossing of $j\omega$-axis also. The plot crosses the $j\omega$-axis at $|G(j\omega) H(j\omega)| = -1.9$ K for $\omega = 6.126$ rad/sec. This is also indicated in the Fig.

From Fig. it is clear that if $8K > 1$ or $K > 0.125$, $(-1, j0)$ point is encircled once in anticlockwise direction and hence

$$N = 1$$

Since $P = 1$

and $N = P - Z$

$$Z = 0$$

$\therefore$ The system is stable for $K > 0.125$.

If $K < 0.125$, the $(-1, j0)$ point is encircled once in the clockwise direction and hence $N = -1$

Since $P = 1$

and $N = P - Z$

$$Z = 2$$

There are two closed loop poles in the RHP and hence the system is unstable.

COMPENSATING NETWORKS

A compensator is a physical device which may be an electrical network, mechanical unit, Pneumatic, hydraulic or a combination of various types of devices. The electrical networks are mostly used. It is easy to design RC filters. For the design of compensation network mainly transfer function approach is used.

There are three compensating networks.

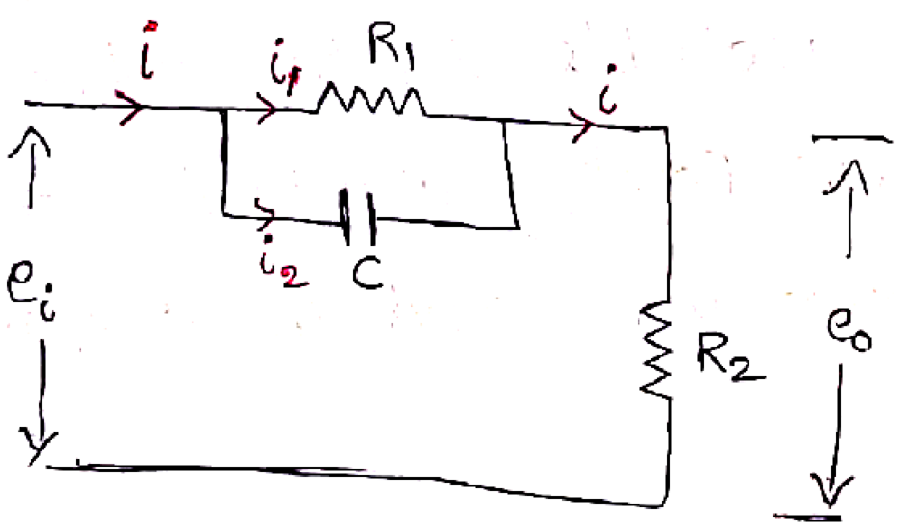
1) Lead network or Lead compensator
2) Lag network or Lag compensator
3) Lag-Lead network or Lag-lead compensator

Lead Network

If a sinusoidal i/p is applied to a network, a sinusoidal steady-state o/p having a ~~phase~~ phase is obtained.

If the steady state o/p leads the i/p, the network is called Lead network.

The lead compensating network is shown in the following fig.



Apply KCL to the circuit

$$i = i_1 + i_2$$

$$\frac{e_o}{R_2} = \frac{e_i - e_o}{R_1} + C\frac{d}{dt}(e_i - e_o)$$

Taking Laplace Transform on both sides

$$\frac{1}{R_2}E_o(S) = \frac{1}{R_1}\left[E_i(S) - E_o(S)\right] + CS\left[E_i(S) - E_o(S)\right]$$

$$= E_i(S)\left(\frac{1}{R_1} + SC\right) - E_o(S)\left(\frac{1}{R_1} + SC\right)$$

$$\Rightarrow \quad E_o(S)\left[\frac{1}{R_1} + \frac{1}{R_2} + SC\right] = E_i(S)\left[\frac{1}{R_1} + SC\right]$$

$$E_o(S)\left[\frac{R_1 + R_2 + SCR_1R_2}{\cancel{R_1}R_2}\right] = E_i(S)\left(\frac{1 + SCR_1}{\cancel{R_1}}\right)$$

$\therefore$ Transfer function

$$\frac{E_o(S)}{E_i(S)} = \frac{R_2(1 + SCR_1)}{R_1 + R_2 + SCR_1R_2}$$

$$= \frac{\cancel{R_1R_2C}\left(S + \frac{1}{R_1C}\right)}{\cancel{R_1R_2C}\left(S + \frac{R_1+R_2}{R_1R_2C}\right)}$$

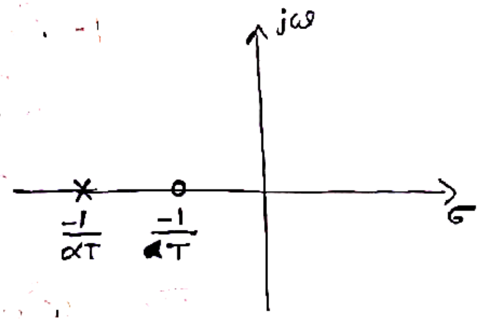
$$= \frac{\left(S + \frac{1}{R_1C}\right)}{S + \frac{1}{\left(\frac{R_2}{R_1+R_2}\right)}\frac{1}{R_1C}}$$

$$\frac{E_o(S)}{E_i(S)} = \frac{S + \frac{1}{T}}{S + \frac{1}{\alpha T}}$$

where $T = R_1C$

$$\alpha = \frac{R_2}{R_1 + R_2} < 1$$

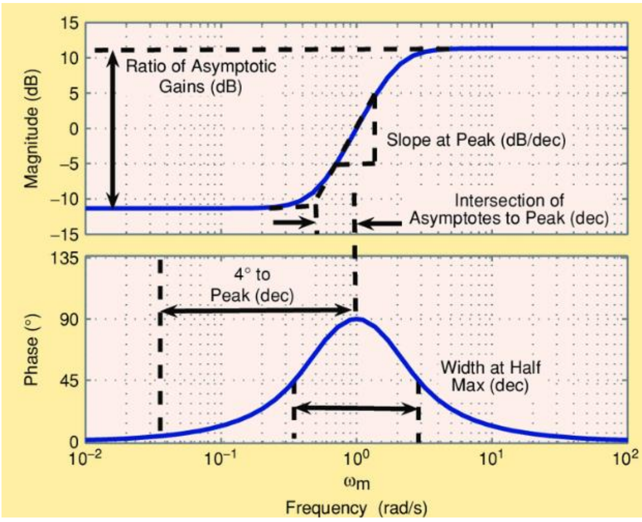
The Pole-zero plot of lead compensator is shown in the fig.



From the Pole-zero plot, the zero is nearer to the imaginary axis as compared to pole.

Since, $0 < \alpha < 1$, the zero is always located to the right of the pole.

The frequency response of lead compensator is shown in the fig.



EFFECTS OF LEAD NETWORK

1) Since a lead compensator adds a dominant zero and Pole, the damping of the closed loop system is increased.

2) Due to the increase of damping the overshoot, rise time and settling time are reduced and hence the transient response can be improved.

3) It improves the Phase Margin of the closed loop system.
4) The velocity constant is usually increased.
5) The slope of the magnitude curve is reduced at the gain cross over frequency, with the result relative stability improve.
6) Bandwidth is increased.
7) The response is faster.
8) The steady state error does not get effected.

LIMITATIONS OF LEAD NETWORK

1) From a single lead n/w, the maximum lead angle available is about 60°. For lead of more than 70° to 90° a multi stage lead compensator is required.

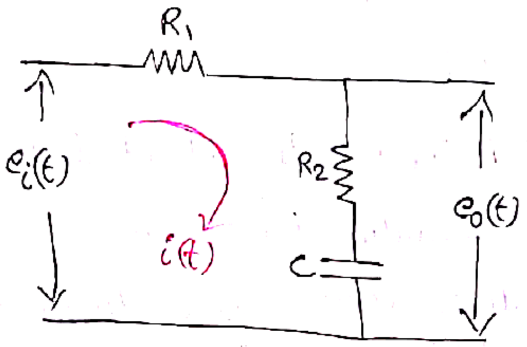
2) Since the compensated system may have a larger undershoot than overshoot, there is tendency to over compensate a system, which may lead to a conditionally stable system.

3) The noise entering the system is more susceptible to the noise signals due to increase in the high frequency gain and hence more bandwidth is sometimes not desirable.

LAG NETWORK

If the steady state o/p lags the i/p, the network is called Lag Network.

The Lag compensating network is shown in the following fig.



Apply KVL to the loop

$$e_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)\,dt$$

taking laplace Transform on both sides

$$E_i(S) = R_1 I(S) + R_2 I(S) + \frac{1}{SC} I(S)$$

$$E_i(S) = \left[R_1 + R_2 + \frac{1}{SC}\right] I(S) \quad \text{--- (1)}$$

& o/p equation is

$$e_o(t) = R_2 i(t) + \frac{1}{C}\int i(t)\,dt$$

Taking laplace Transform on both sides

$$E_o(S) = R_2 I(S) + \frac{1}{SC} I(S)$$

$$E_o(S) = \left[R_2 + \frac{1}{SC}\right] I(S) \quad \text{--- (2)}$$

$$\frac{eq.\ 2}{eq.\ 1} \Rightarrow \quad \therefore \text{ Transfer function } = \frac{E_o(S)}{E_i(S)}$$

$$= \frac{R_2 + \frac{1}{SC}}{R_1 + R_2 + \frac{1}{SC}} = \frac{1 + R_2 SC}{1 + (R_1 + R_2)SC}$$

$$= \frac{R_2 \cancel{C}}{(R_1 + R_2)\cancel{C}} \cdot \frac{S + \frac{1}{R_2 C}}{S + \frac{1}{(R_1 + R_2)C}}$$

$$\therefore \text{ Transfer function } = \frac{E_o(S)}{E_i(S)}$$

$$= \frac{R_2}{R_1 + R_2} \; \frac{S + \frac{1}{R_2 C}}{S + \frac{1}{(R_1 + R_2)C}}$$

let $T = R_2 C$

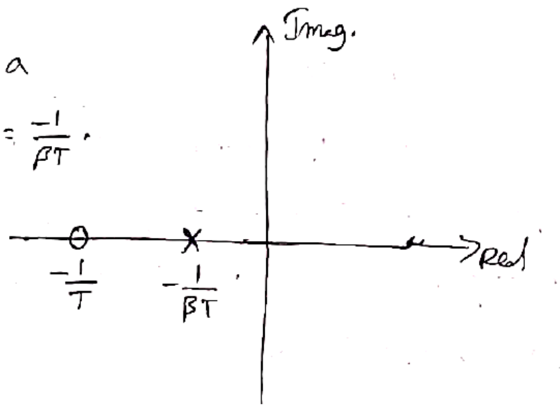
& $\beta = \dfrac{R_1 + R_2}{R_2} > 1$

[Generally $\beta = 10$]

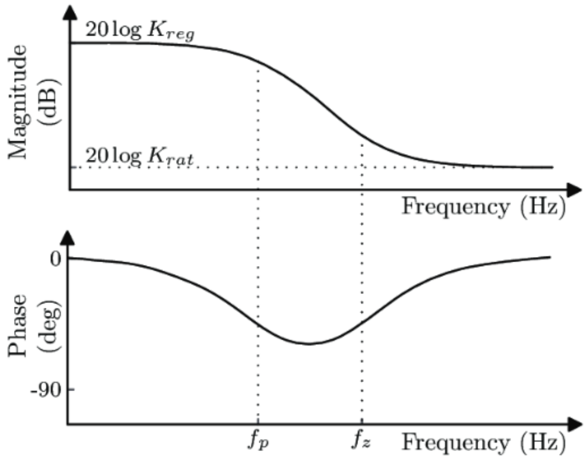
$$\boxed{\frac{E_o(S)}{E_i(S)} = \frac{1}{\beta} \; \frac{S + \frac{1}{T}}{S + \frac{1}{\beta T}}}$$

The Pole-Zero Plot of lag compensator is shown in the following fig.

The lag compensator has a zero at $s = -\frac{1}{T}$ and a pole at $s = \frac{-1}{\beta T}$. Since $\beta > 1$, the pole is always located to the right of the zero.



The frequency response of lag compensator is shown in the fig.



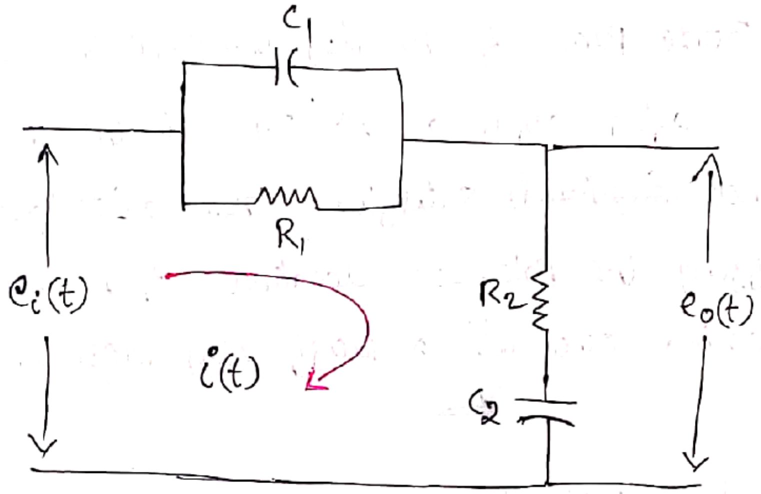
Effects of Lag Network

1) For a given relative stability, the velocity constant is increased.
2) There is decrease in gaincrossover frequency, thus decreasing the bandwidth.
3) PM increases.
4) Response will be slower.
5) Rise time & settling time become large.

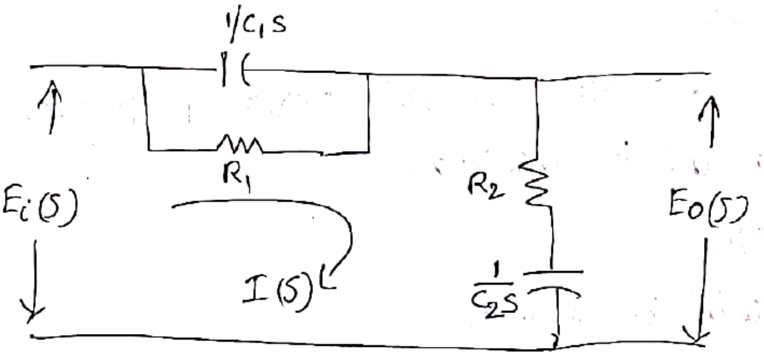
Lead-Lag Network

The lead-Lag compensating Network is used to improve the speed of the response and the steady state error.

The lead-lag network can be shown in the fig.



Apply Laplace Transform to the circuit and the circuit is redrawn as shown in the fig.



Apply KVL to the circuit

$$E_i(S) = I(S)\left(\frac{R_1 \cdot \frac{1}{C_1 S}}{R_1 + \frac{1}{C_1 S}} + R_2 + \frac{1}{C_2 S}\right)$$

$$= I(S)\left(\frac{\frac{R_1}{\cancel{C_1 S}}}{\frac{1 + SC_1R_1}{\cancel{C_1 S}}} + R_2 + \frac{1}{C_2 S}\right)$$

$$E_i(S) = I(S)\left[\frac{R_1}{1+SC_1R_1} + R_2 + \frac{1}{C_2 S}\right]$$

$$E_i(S) = I(S)\left(\frac{SC_2R_1 + R_2C_2S(1+SC_1R_1) + (1+SC_1R_1)}{(1+SC_1R_1)(C_2S)}\right)$$

$$E_i(S) = I(S)\left(\frac{SC_2R_1 + SR_2C_2 + S^2C_1C_2R_1R_2 + SC_1R_1 + 1}{(1+SC_1R_1)\,C_2S}\right)$$

The O/P voltage $E_o(S) = I(S)\left[R_2 + \frac{1}{C_2 S}\right]$

$$E_o(S) = I(S)\left(\frac{1+SC_2R_2}{C_2S}\right)$$

$\therefore$ Transfer function, $\frac{E_o(S)}{E_i(S)}$

$$\frac{E_o(S)}{E_i(S)} = \frac{(1+SC_1R_1)(1+SC_2R_2)}{S^2C_1C_2R_1R_2 + S[R_1C_1 + R_2C_2 + R_1C_2] + 1}$$

$$= \frac{\cancel{R_1C_1R_2C_2}\left[s+\frac{1}{R_1C_1}\right]\left[s+\frac{1}{R_2C_2}\right]}{\cancel{R_1R_2C_1C_2}\left[s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}\right]}$$

Let $T_1 = R_1C_1$ & $T_2 = R_2C_2$

$$\frac{1}{\alpha T_1} + \frac{1}{\beta T_2} = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}$$

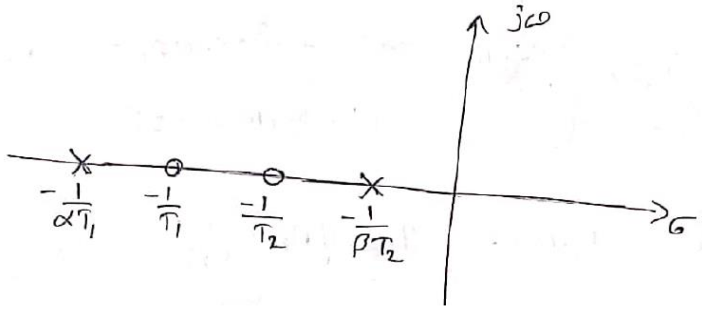
$$\alpha\beta T_1T_2 = R_1R_2C_1C_2 \qquad \left[\because \alpha\beta = 1,\ \beta > 1\right]$$

$\therefore$ Transfer function

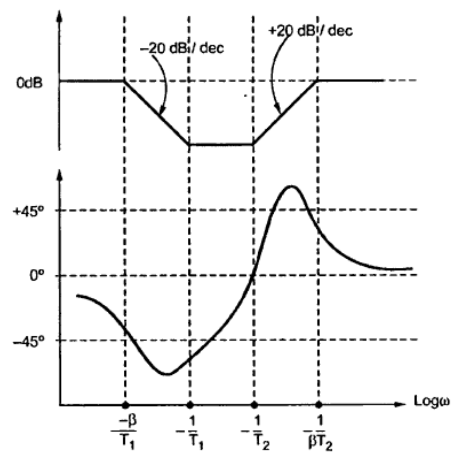
$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}}$$

The phase lead portion adds the phase lead angle and the phase lag portion provides attenuation near and above the gain cross over frequency.

The pole-zero plot for lead-lag compensator can be shown below.



The frequency response of lead-lag compensator is shown in the fig.



Design Steps of Lead Compensator using Bode plot

Step 1: The open loop gain K of the given system is determined to satisfy the requirement of the error constant.

Step 2: After determining the value of K, draw bode plot of uncompensated system.

Step 3: The phase margin of the uncompensated system determined from the bode plot.

Step 4: Determine the amount of phase angle to be contributed by the lead network by using formula given below:

$$\phi_m = \gamma_d - \gamma + \zeta$$

where, $\phi_m \rightarrow$ Maximum phase lead angle

$r_d \rightarrow$ Desired phase margin

$\gamma \rightarrow$ Phase margin of uncompensated system

$\zeta \rightarrow$ Additional phase lead to compensate for shift in gain crossover frequency.

Choose an initial choice of ζ as $\pm 5°$

Note: If ϕ_m is more than 60° then realize the compensator as cascade of two lead compensator with each compensator contributing half of the required angle.

Step 5: Determine the transfer function of lead compensator. Calculate α using the equation,

$$\alpha = \frac{1 - \sin \phi_m}{1 + \sin \phi_m}$$

From the bode plot, determine the frequency at which the magnitude of $G(j\omega)$ is $-20 \log \frac{1}{\sqrt{\alpha}}$ *db*.

This frequency is ω_m

Calculate T,

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

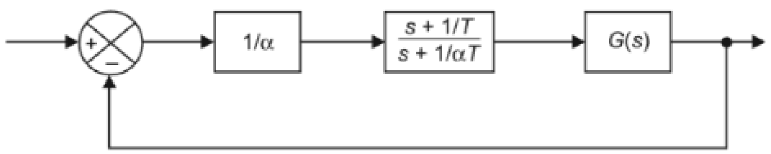
$$\therefore \quad T = \frac{1}{\omega_m \sqrt{\alpha}}$$

Transfer function of lead compensatio n,

$$G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \frac{\alpha(1 + sT)}{(1 + \alpha s T)}$$

Step 6: Determine the open loop transfer function of compensated system:

The lag compensator is connected in series with $G(s)$ as shown in Fig. When the lead network is inserted in series with the plant, the open loop gain of the system is attenuated by the factor α ($\therefore$ $\alpha < 1$), so an amplifier with the gain of $1/\alpha$ has to be introduced in series with the compensator to nullify the attenuation caused by the lead compensator.



Open loop transfer function of the overall system

$$G_o(s) = \frac{1}{\alpha}\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}\cdot G(s)$$

$$= \frac{1}{\alpha}\cdot\frac{\alpha(1+sT)}{(1+s\alpha T)}G(s) = \frac{(1+sT)}{(1+s\alpha T)}\cdot G(s)$$

Step 7: Verify whether it satisfies the given specifications. If the phase margin of the compensated system is less than the required phase margin then repeat step 4 to 10 by taking $\in$ as 5° more than previous design.

Design Problem:

Design a cascade compensation for a system whose transfer function is

$$G(s) = \frac{K}{s(1+0.1s)(1+0.001s)}$$

It will fullfill the following specifications

Phase margin ≥ 45°

Velocity constant K_v = 1000 sec^{-1}

SOL:

$$K_v = \lim_{s\to 0} s.G(s) = \lim_{s\to 0} s\cdot\frac{K}{s(1+0.1s)(1+0.001s)}$$

$$K_v = K$$

$$K = 1000$$

$$G(s) = \frac{1000}{s(1+0.1s)(1+0.001s)}$$

Step 2 : Draw the Bode's plot for the transfer function

Two corner frequencies are 1/0.1 = 10 rad/sec. and 1/0.001 = 1000 rad/sec.

ω	Arg(1000) ϕ_1	– Arg(0 + jω) ϕ_2	– Arg(1 + 0.1jω) ϕ_3	– Arg(1 + j0.001ω) ϕ_4	Resultant $\phi_1+\phi_2+\phi_3+\phi_4$
1	0	- 90°	- 5.7°	- 0.06	- 95.7°
5	0	- 90°	- 26.5°	- 0.28	- 116.5°
10	0	- 90°	- 45°	- 0.57	- 135.63°
50	0	- 90°	- 78.6°	- 2.86°	- 171.46°
100	0	- 90°	- 84.2°	- 5.71°	- 179.9°
150	0	- 90°	- 86.2°	- 8.5°	- 184°
200	0	- 90°	- 87.13°	- 11.3°	- 188.43°
500	0	- 90°	- 88.85°	- 26.56°	- 205.41°

Step 3 : From Bode plot :

Phase margin avilable $\phi = 0°$

Specified phase margin $\phi_s = 45°$

Margin of safety $\varepsilon = 5°$

$$\therefore \quad \phi_m = 45° - 0 + 5° = 50°$$

Step 4 : Calculation of 'a'

$$\sin\phi_m = \frac{a-1}{a+1}$$

$$\sin 50 = \frac{a-1}{a+1}$$

$$\therefore \quad a = 7.51$$

Step 5 : Calculation of ω_m

Zero frequency attenuation $= -10\log a$

$= -10\log 7.51 = -8.75$ db

At the gain of -8.75 *db* draw a line on magnitude curve, this will gives ω_m (new gain cross over frequency).

$\therefore$ $\omega_m = 170$ rad/sec. (from Bode plot)

Step 6 : Calculation of 'T'

$$\omega_m = \frac{1}{T\sqrt{a}}$$

$$a = 7.51$$

$$\omega_m = 170$$

$$\therefore \quad T = 0.00214$$

Step 7 : Transfer Function of Compensator

$$\boxed{G_c(s) = \frac{1}{7.51}\left(\frac{1+0.016s}{1+0.00214s}\right)}$$

The amplification necessary to cancel the lead network attenuation of 7.51

$$\therefore \quad G_c(s) = \frac{1+0.016s}{1+0.00214s}$$

Step 8 : Overall Transfer Function

$$G(s) = G(s) \,.\, Gc\,(s)$$

$$= \frac{1000(1+0.016s)}{s(1+0.1s)\,(1+0.001s)\,(1+0.00214s)}$$

Step 9 : Draw the Bode plot of overall transfer function & check

The corner frequencies are

$\omega_1 = 10$ rad/sec.

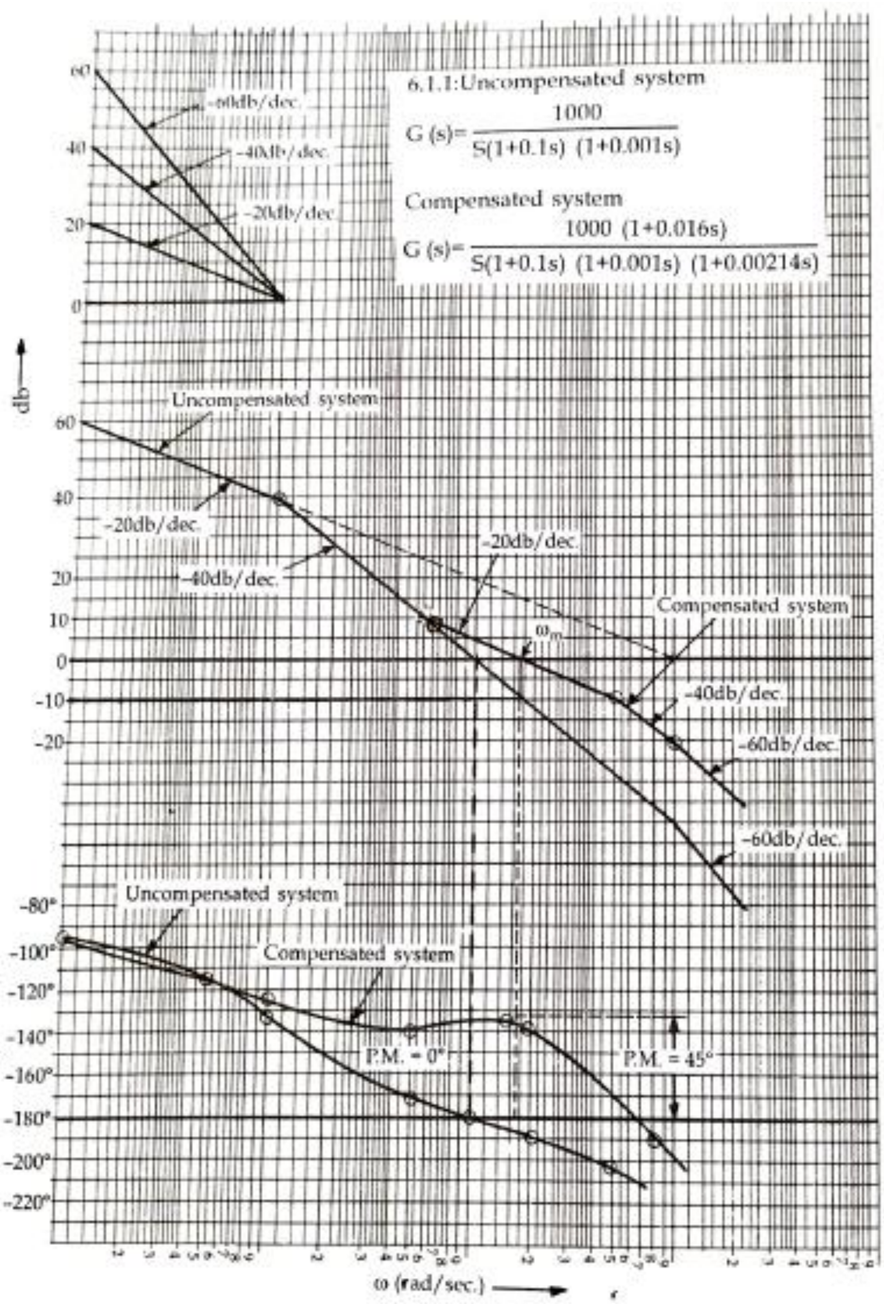
$\omega_2 = 62.5$ rad/sec.

$\omega_3 = 476.5$ rad/sec.

$\omega_4 = 1000$ rad/sec.

ω	Arg 1000 ϕ_1	-Arg (jω) ϕ_2	-Arg (1+j0.1ω) ϕ_3	-Arg (1+j0.001ω) ϕ_4	-Arg (1+j0.002ω) ϕ_5	-Arg (1+j0.016ω) ϕ_6	Resultant $\phi_1+\phi_2+\phi_3+\phi_4+\phi_5+\phi_6$
1	0	-90°	-5.7°	0.05	0.11	0.91	-95°
10	0	-90°	-45°	0.57	1.14	9	-126°
50	0	-90°	-78.6°	-2.86°	-6°	38.65°	-138.8°
100	0	-90°	-84.2°	-5.71°	-11.85°	57.99°	-133.77°
150	0	-90°	-86.1°	-8.5°	-17.4°	63.38°	-134.6°
800	0	-90°	-89.2°	-38.65°	-59.23°	85.5°	-191.58°
200	0	-90°	-87.13°	-11.3°	-22.78°	72.6°	-138.61°

From Bode's plot of copensated system P.M. = 45°



Design Steps of Lag Compensator using Bode plot

Step 1 : The magnitude and phase *Vs* frequency curves (Bode plot using asymptotic approximation) are plotted for $G(s)$ of the uncompensated system, with gainconstant K set according to steady state error requirement.

Step 2 : From the Bode plot, determine the phase margin of the uncompensated system.

Step 3 : If ϕ_s = specified phase margin

ε = marign of safety

$\phi = \phi_s + \varepsilon$.

Step 4 : Determine the frequency corresponding to the required phase margin from the phase curve. This frequency is new gain cross over frequency (ω'_m)

Step 5 : The magnitude curve is brought down to 0 db at the new gain cross-over frequency where the phase margin is satisfied, the phase lag network must provide the amount of attenuation equal to the value of magnitude curve at ω'_m

$$|G(j\omega'_m)| = -20 \log a \qquad a < 1$$

or, $$a = 10^{-|G(j\omega'_m)|/20} \qquad a < 1$$

calculate 'a' from above expression.

Step 6 : Calculate 'T' from

$$\frac{1}{aT} = \frac{\omega'_m}{10}$$

usually the upper corner frequency $(1/aT)$ is placed at a frequency about one decade below the new gain-cross over frequency.

Step 7 : Draw the Bode's plot for compensated network & investigate to see if the required phase margin is met or not, if not, adjust the value of 'a' & 'T'.

Design Steps of Lag-Lead Compensator using Bode plot

Step 1: Determine the openloop gain K of the uncompensated system to satisfy specified error requirement.

Step 2: Draw the bode plot of uncompensated system

Step 3: From the bode plot determine the gain margin of the uncompensated system.

Let, ϕ_{gc} = Phase of $G(j\omega)$ at gain cross over frequency.

$\gamma \Rightarrow$ Phase margin of uncompensated system.

Now, $\gamma = 180° + \phi_{gc}$

If the gain margin is not satisfactory then compensation is required.

Step 4: Choose a new phase margin

Let, γ_d = Desired phase margin

Now, new phase margin,

$$\gamma_n = \gamma_d + \in$$

Choose an initial value of $\in = \pm 5°$

Step 5: From bode plot, determine the new gain cross over frequency, which is the frequency corresponding to a phase margin of $\gamma_{n.}$

Let,

$$\omega_{gcn} = \text{New gain cross over frequency}$$

and $$\phi_{gcn} = \text{Phase of } G(j\omega) \text{ at } \omega_{gcn}$$

$$\gamma_n = 180^o + \phi_{gcn}$$

or $$\phi_{gcn} = \gamma_{n.} - 180^o$$

In the phase plot of uncompensated system, the frequency corresponding to a phase of ϕ_{gcn} is the new gain crossover frequency ω_{gcn}. Choose the gain crossover frequency of the lag compensator, ω_{gcl}, some what greater than ω_{gcn} (i.e. choose ω_{gcl} such that $\omega_{gcl} > \omega_{gcn}$) .

Step 6: Calculate β of Lag compensator.

Let, $$A_{gcl} = |G(j\omega)| \text{ in } db \text{ at } \omega = \omega_{gcl}$$

From the bode plot find A_{gcl}

$$[\, A_{gcl} / 20 \,]$$

Now, $$A_{gcl} = 20 \log \beta$$

or $$\beta = 10^{(A_{gcl}/20)}$$

Step 7: Determine the transfer function of Lag section.

The zero of the lag compensation is placed at a frequency one-tengthof ω_{gcl}

$\therefore$ zero of lag compensator,

$$z_{c1} = 1/\, T_1 = \omega_{gcl10}$$

Now, $$T_1 = 10/\, \omega_{gcl}$$

Pole of lag compensator,

$$p_{c1} = \frac{1}{\beta T_1}$$

Transfer function of lag section

$$G_1(s) = \frac{(s + 1/T_1)}{(s + 1/\beta T_1)} = \beta \frac{(1 + sT_1)}{(1 + s\beta T_1)}$$

Step 8: Determine the transfer function of lead section.

Take, $$\alpha = 1/\beta$$

From the bode plot find ω_m which is the frequency at which the db gain is $-20 \log\, (1/\sqrt{\alpha})$.

Now $$T_2 = 1/\,\omega_m \sqrt{\alpha}$$

Transfer function of lead section

$$G_2(s) = \frac{(s + 1/T_2)}{(S + 1/\alpha T_2)} = \alpha \frac{(1 + sT_2)}{(1 + s\alpha T_2)}$$

Step 9: Determine the tansfer function of lag lead compensator.

Transfer function of lag-lead compensator,

$$G_c(s) = G_1(s) \times G_2(s)$$

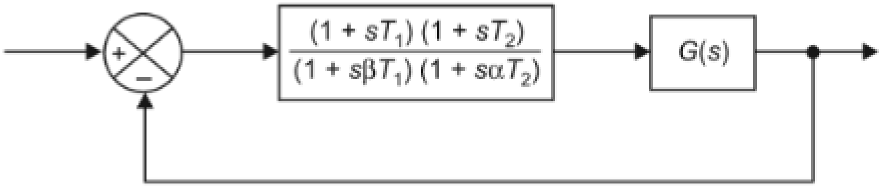
$$= \beta \frac{(1 + sT_1)}{(1 + s\alpha T_1)} \times \alpha \frac{(1 + sT_2)}{(1 + s\alpha T_2)}$$

Since $\alpha = 1/\beta$

$$G(s) = \frac{(1 + sT_1)}{(1 + s\beta T_1)} \cdot \frac{(1 + sT_2)}{(1 + s\alpha T_2)}$$

Step 10: Determine the open loop transfer function of compensated system.

The lag lead compensator is connected in series with $G(s)$ as shown in Fig.



Open loop transfer function of compensated system

$$G_o(s) = \frac{(1 + sT_1)(1 + sT_2)}{(1 + s\beta T_1)(1 + s\alpha T_2)} \times G(s)$$

Step 11: Draw the plot of compensated system and verify whether the specifications are satisfied (or) not. If the specifications are not satisfied then choose another choice of $\alpha \angle 1/\beta$ and repeat the step 8 to 11 till design get satisfied.